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5 Section E

E1

What is the 41st odd whole number above 41?

Solution. The 1st odd whole number above 41 is 43. The second is 45. Following this pattern, in general the nth odd whole number above 41 is $41 + 2n$. Substituting $n = 41$, we get $41 + 2 \times 41 = 3 \times 41 = 123$.

Answer to E1: 123

E2

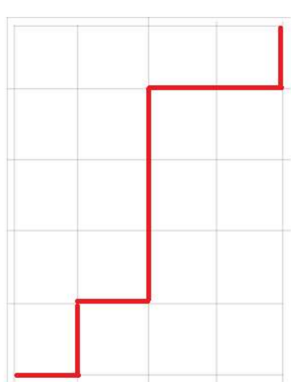
In an isosceles trapezoid, the non-parallel sides both have length 5. The smaller of the parallel sides has length 6 and the trapezoid has a height of 4. What is its area?

Solution. Draw lines from the two endpoints on the shorter parallel side perpendicular to the longer parallel side. This splits our trapezoid into two triangles and a rectangle. The area of the rectangle is $6 \times 4 = 24$. For the triangles, since they are right angled, by Pythagorean Theorem the bases have length $\sqrt{5^2 - 4^2} = 3$. Then each triangle has area $\frac{1}{2} \times 3 \times 4 = 6$. Then the total area of the trapezoid is $24 + 2 \times 6 = 36$.

Answer to E2: 36

E3

Jay is walking around in a 5×4 grid. He starts at the bottom left corner and walks to the top right corner. At each step, he can either walk one space to the right, or one space upward. One such path is shown below.



How many paths can Jay take which do not pass through the top left or bottom right corners of the grid?

Solution. We can represent steps north by the letter N and steps east by the letter E. This becomes a combinatorics problem where I want the number of ways to arrange 5 N's among $5+4=9$ places. This is given by $\binom{9}{5} = \frac{9\times8\times7\times6}{4\times3\times2} = 126$. There is exactly one path that passes through the top left (all N then all E), and one that passes through the bottom right. Then there are $126-1-1=124$ paths.

Answer to E3: 124

E4
Given $2n$ points spaced evenly on the circle, a perfect matching is a set of n straight lines that connect each point to exactly one other point. What is the number of perfect matchings of 8 points around a circle?

Solution. Pick one point to start with. We can choose one of 7 other points to connect it with. Pick a point that hasn't been connected yet. There are 5 choices on how to connect it. Picking another point that hasn't been connected, there are 3 choices. Finally, only two points remain, and so there is 1 way to connect them. This gives $7\times5\times3=105$ perfect matchings.

Answer to E4: 105

E5
If three line segments whose lengths are integers $x \le y \le z$ can form a triangle with perimeter 9, what is the probability that this triangle is scalene?

Solution. We start by listing the ways in which we can partition 9 as a sum of three integers (in decreasing order).

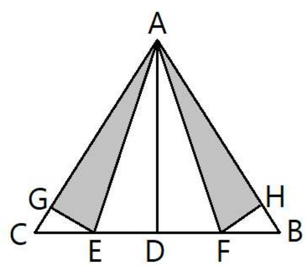
$9=7+1+1$
$9=6+2+1$
$9=5+3+1$
$9=5+2+1$
$9=4+4+1$
$9=4+3+2$
$9=3+3+3$

In order for these to form a triangle, they have to satisfy the triangle inequality, $x + y > z$. Of these seven partitions, only the last three satisfy it. Only one of these is scalene, so the probability is $\frac{1}{3}$.

Answer to E5: $\frac{1}{3}$

E6

In the following figure, ABC is an equilateral triangle, D is the midpoint of BC, E is the midpoint of CD, F is the midpoint of DB, EG is perpendicular to AC, and FH is perpendicular to AB. What fraction of the triangle is shaded?



Solution. Suppose that the equilateral triangle has side length 1. Then $AD = \frac{\sqrt{3}}{2}$. Since EG is perpendicular to AC, $\angle CGE = 90^\circ$. Also, $\angle GCE = \angle ACD$ since they are really the same angle. But ADC is a right angled triangle as well, so all three angles in ADC and GCE are equal, hence they are similar triangles. We have that $CE = \frac{1}{4}$, while $AC = 1$. Since the ratio between the sides is $4:1$, the ratio between their areas is $16:1$.

The area of triangle ADC is $\frac{1}{2} \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$. Then the area of triangle CGE is $\frac{1}{16} \times \frac{\sqrt{3}}{8} = \frac{\sqrt{3}}{128}$. Finally, the area of ADE is $\frac{1}{2} \times \frac{1}{4} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{16}$. Then the area of AGE is $\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{16} - \frac{\sqrt{3}}{128} = \frac{7\sqrt{3}}{128}$. The fraction of ABC that is shaded is the same as the fraction of ADC that is shaded by symmetry, and so we get that $\frac{7\sqrt{3}}{128} / \frac{\sqrt{3}}{8} = \frac{7}{16}$.

Answer to E6: $\frac{7}{16}$

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E7
Alice and Bob are playing a game, Alice goes first. They take turns placing one piece (of their color) down on an empty spot of an infinitely large grid. On Bob’s first move, he is not allowed to place a piece diagonally adjacent to Alice’s first piece. Whoever makes a row, column, or diagonal of 4 pieces (of their color) in a row first wins. Assuming Alice and Bob play perfectly, what is the minimum number of moves needed to for one of them to win?

Solution. How can Alice force a win in this game? If she has three pieces in a row, and Bob is not blocking in either direction, he can only block one of the two ways she can connect a fourth piece, and so she wins. Notice that if she makes an ”L” shape with three pieces which is not blocked, she is threatening to create two different unblocked groups of three pieces in a row. Bob can only block one of these, so she still wins. So Alice should try to force this shape. Also notice that Alice wins any ”race” to 4 where both players ignore each other, so Bob is best served placing a piece close to one of Alice’s.

Suppose Alice places a piece somewhere, and then Bob places a piece above hers (by symmetry, the other three directions will have the same outcome). Alice should then place a piece to the left or right of Bob’s, threatening a diagonal of three in a row, and Bob cannot block this threat in either direction while putting a piece in line with his first piece. He must block the threat directly. If he does not, Alice can place a third piece on the diagonal, farthest away from Bob’s pieces, and then win. Once Bob blocks this, Alice can place a piece adjacent to both of the pieces she has placed so far. This creates the ”L” shape that she mentioned before, which is unstoppable since Bob has no pieces in line with either of Alice’s lines. Bob blocks one of the threats, and then Alice extends her other line to create three in a row with neither end blocked. No matter what Bob does, she wins on her next move. Between the two of them, they have made 9 moves.

Answer to E7: 9

E8

A cellular automaton is a machine that reads in a strip of squares, each of which contains 1 or 0. The machine computes a new number for each square based on the input it receives. A square on an edge is kept the same. A square not on an edge is given a value according to the values of the squares before and after it, as well as the current value of the square itself. For example, suppose we have the following table summarizing how the machine works:

Input	111	110	101	100	011	010	001	000
Output	0	1	0	1	1	1	0	0

Here is an example of an input and output with this machine.

0	1	0	0	1	0	1	1	1	0
0	1	1	0	1	0	1	0	1	0

The third entry of the output strip is a 1 since the entries above it from left to right in the input strip are 1, 0, and 0, so we look up 100 in the table and see that it generates a 1.

If we take the binary number corresponding to the Output row of the machine's table, 01011100, it corresponds to the decimal number

$$0 \times 2^7 + 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 92$$

We say that the machine was **following rule 92**.

Suppose that a different cellular automaton receives as input the strip

0	1	1	1	0	1	0	0	0	1

and outputs the strip

0	0	0	1	1	0	1	0	1	1

What rule was the machine following?

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Solution. The input and output give us enough information to produce a table like the one in the statement of the question.

Input	111	110	101	100	011	010	001	000
Output	0	1	1	1	0	0	1	0

The binary number corresponds to the decimal number $2^6+2^5+2^4+2^1 = 114$. So the machine was following rule 114.

Answer to E8: 114