Student Name: ______________________________

Please write your name on *every* page.

6 Section F

F1

A math professor curves his class according to a ***square root curve***: he takes a student's raw score of $X\%$ on an exam, and curves it to $10\sqrt{X}\%$ (so that 0% goes to 0% and 100% goes to 100%). If the difference between a student's score before and after the curve is 16%, and the student scored at least 50% before the curve, what was their original score?

Solution. We have that $10\sqrt{X} - X = 16$. Note that we may write $X = \sqrt{X}^2$, and so $\sqrt{X}^2 - 10\sqrt{X} + 16 = 0$. This factors as $(\sqrt{X} - 8)(\sqrt{X} - 2) = 0$, so $\sqrt{X} = 8$ or $\sqrt{X} = 2$. Thus $X = 64$ or $X = 4$. Since X is the original score, and it is at least 50%, it must be 64%.

Answer to F1: 64%

F2

In a classroom, if all boys get into groups of 2 and all girls get into groups of 3, there are 32 total groups. If instead all boys get into groups of 5 and all girls get into groups of 2, there are 26 total groups. How many students are in the class?

Solution. Let b be the number of boys in the class and g be the number of girls. We are given that $b/2 + g/3 = 32$ and $b/5 + g/2 = 26$. Clearing the fractions, $3b + 2g = 192$ and $2b + 5g = 260$. Then $11(b + g) = 3(3b + 2g) + (2b + 5g) = 3 \times 192 + 260 = 836$, so $b + g = 836/11 = 76$. Thus there are 76 students in the class.

Answer to F2: 76

F3

A degree 3 polynomial $x^3 + bx^2 + cx + d$ is such that b and c are integers between 0 and 9, and such that $x = 10$ is a root. What is the largest possible value of $|d|$?

Solution. If $x = 10$ is a root, we have that $10^3 + b \times 10^2 + c \times 10 + d = 0$, or $10^3 + b \times 10^2 + c \times 10 = -d$. Since b and c are between 0 and 9, this is the representation of a base 10, four digit number with digits $1, b, c, 0$ in order. To maximize this, we want b and c to be 9, in which case $|d| = 1990$.

Answer to F3: 1990

Student Name: ______________________________
Please write your name on ***every*** page.

F4
Initially, there are two amoebas in Amoebaland. At the end of each day (starting with day 1), each amoeba tries to divide into more amoebas. For each amoeba, its success rates are as follows:

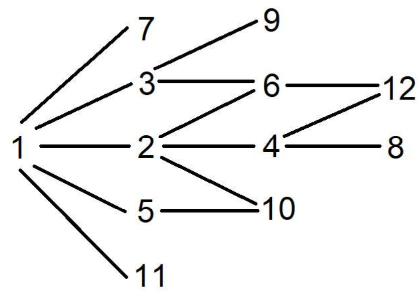
- With probability $\frac{1}{4}$, it fails (no new amoeba are born)
- With probability $\frac{1}{2}$, it divides into two amoeba (so one new amoeba is born)
- With probability $\frac{1}{4}$, it divides into three amoeba (so two new amoeba are born)

What is the expected number of amoeba in Amoebaland at the beginning of day 3?

Solution. The key to this problem is the symmetry. We could compute all the probabilities, but this would be tedious and this is a timed exam, so any shortcuts are welcome. Instead, notice that the distribution is symmetric about the mean. That is, there is an equal chance of being the same distance below the mean and above. Indeed, computing the probabilities for the first division, we get $1/16, 1/4, 3/8, 1/4, 1/16$ chances for $2, 3, 4, 5,$ and 6 amoebas respectively. The same idea applies to this new distribution, and so forth, where we always have symmetry. Then our expected value is simply the mean. This occurs when everything divides to create one new amoeba each day, so at the end of day 1 we have 4, and at the end of day 2 we have 8.

Answer to F4: 8

F5
We say that x is a proper divisor of y if y is divisible by x and $x \neq y$. A tree of the numbers from 1 to n is constructed by drawing a line between x and y precisely when x is a proper divisor of y, and there is no z such that x is a proper divisor of z and z is a proper divisor of y. For example, here is the tree for numbers 1 to 12.



How many lines does the tree for numbers 1 to 50 contain?

Solution. If $a < b$ and a is connected to b in the tree, then we must have that $b = ap$ for some prime p. To see this, suppose not. Since they are connected, a is a proper divisor of b, and so $b = ax$ for some x not prime. Write $x = pq$ as a non-trivial factorisation (with neither p nor q being 1). But then ap is an intermediary proper divisor, with a being a proper divisor of ap and ap being a proper divisor of $apq = ax = b$. But then a and b are not connected in the tree, a contradiction.

Using this observation, we can reason that the predecessors of x in the tree are obtained by dividing out a single copy of one of the prime factors of x. For example, with $12 = 2^2 \times 3$ we can divide out by 2 to get 6 or 3 to get 4. The image we are given confirms this. Thus the question reduces to counting the number of numbers with a certain number of distinct prime factors.

- The only number with zero prime factors is 1.
- A number with one distinct prime factor is of the form p^k for some prime p. With 50 as an upper bound, we get that $2, 2^2, 2^3, 2^4, 2^5$ work for 2 but $2^6 = 64 > 50$, $3, 3^2, 3^3$ work for 3 but $3^4 = 81 > 50$, $5, 5^2$ and $7, 7^2$ work, and then only the primes themselves work from 11 onwards, since $11^2 = 121 > 50$. These primes are 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. In total this gives us $5 + 3 + 2(2) + 1(11) = 23$ numbers.
- The smallest number with three distinct prime factors is $2(3)(5) = 30$. Multiplying this by 2 gives $60 > 50$, so none of these multiples work. Also, $2(3)(7) = 42$, and again no multiples of this work. Finally $2(3)(11) = 66 > 50$ and $3(5)(7) = 105 > 50$ do not work, and clearly using even larger primes will not work either. So there are 2 such numbers.

It is much harder to count numbers with two distinct prime factors, but we may do this using the work we have already done. No number from 1 to 50 contains four or more distinct prime factors, since the smallest such number is $2(3)(5)(7) = 210 > 50$. Thus the numbers with two distinct prime factors are simply the ones that do not fall into any of the three above categories. This is a total of $50 - 1 - 23 - 2 = 24$.

Now any number has a number of lines to its left equal to the number of distinct prime factors it has. So in total, the diagram will have $23(1) + 24(2) + 2(3) = 77$ lines.

Answer to F5: 77

F6

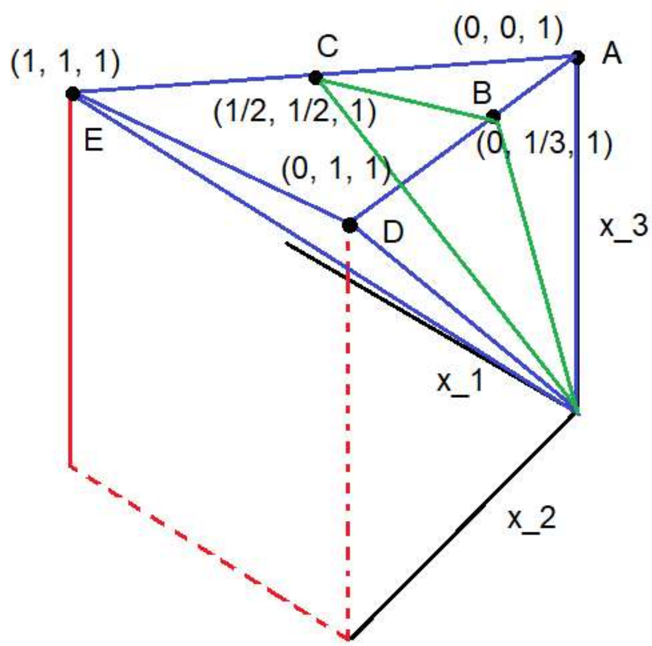
Three numbers are randomly and independently chosen from the unit interval (between 0 and 1). What is the probability their median is less than $\frac{3}{4}$ of their mean?

Student Name: ____________________

Please write your name on *every* page.

Solution. Even though it may not look like it, this is secretly a question about geometric probability. Let the three numbers be x_1, x_2, and x_3, and without loss of generality let $x_1 < x_2 < x_3$. Then x_2 is the median, while the mean is $\frac{1}{3}(x_1 + x_2 + x_3)$. If the median is less than 3/4 of the mean, we have that $x_2 < \frac{3}{4} \cdot \frac{1}{3}(x_1 + x_2 + x_3) = \frac{1}{4}(x_1 + x_2 + x_3)$. Then $3x_2 < x_1 + x_3$ and so $x_2 < \frac{1}{3}(x_1 + x_3)$. Note that the inequalities $x_1 < x_2 < x_3$ and $x_2 < \frac{1}{3}(x_1 + x_3)$ both form tetrahedra in 3D coordinates x_1, x_2, x_3, with each coordinate being bounded below by 0 and above by 1. We will find the areas of these two tetrahedra.

Letting x_3 be the height, at $x_3 = 1$ the equation $3x_2 = x_1 + x_3$ becomes $3x_2 = x_1 + 1$. At $x_1 = 0$ we have $x_2 = 1/3$, so we intersect the axis at $(0, 1/3, 1)$. This same line intersects the line $x_2 = x_1$ where $3x_2 = x_2 + 1$, and so $2x_2 = 1$, meaning $x_2 = x_1 = 1/2$. We get the following diagram.



Since the volume of a pyramid is $bh/3$ where b is the area of the base and h is the height, and since both pyramids have a height of 1, the ratio between their volumes will simply be the ratio between the area of their bases. We have that $[ADE] = \frac{1}{2}(1)(1) = \frac{1}{2}$ since it is a right angled triangle. In contrast, triangle ABC has height 1/2 and base 1/3, so it has area $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$. Thus the ratio between the areas, and hence the probability, is $\frac{1}{2} / \frac{1}{12} = \frac{1}{6}$.

Student Name: ________________________________

Please write your name on *every* page.

Answer to F6: $1/6$

F7

Let $\varphi(n)$ denote the number of positive integers $m \leq n$ for which $\gcd(m, n) = 1$ (that is, m and n share no common factors other than 1). For k a positive integer, let $\varphi_k(n)$ denote $\varphi(\varphi(\cdots(\varphi(n))\cdots))$ where the function φ is applied k times. For example, $\varphi(10) = 4$ since 1, 3, 7, and 9 are the only numbers below 10 which have no factors in common with 10 other than 1, while $\varphi_2(10) = \varphi(\varphi(10)) = \varphi(4) = 2$, since 1 and 3 are the only numbers below 4 which have no factors in common with 4 other than 1, and $\varphi_3(10) = \varphi(2) = 1$, since 1 is the only factor less than or equal to 2 which has no factors in common with 2 other than 1 itself.

What is the largest k such that $\varphi_k(1000) \neq 1$?

Solution. We have that $1000 = 2^3 \times 5^3$. To count the number of numbers with no factors in common with 1000 other than 1, we may instead count the *complement*, that is, the number of numbers with at least one non-trivial factor in common. Clearly such a number must contain a factor of 2 or a factor of 5. There are $1000/2 = 500$ factors of 2 and $1000/5 = 200$ factors of 5. We have double-counted numbers with both a factor of 2 and a factor of 5. There are $1000/10 = 100$ such numbers. This gives us a total of $500 + 200 - 100 = 600$ numbers not coprime with 1000, which gives us $1000 - 600 = 400$ numbers coprime with 1000, so $\varphi(1000) = 400$. Similarly,

$\varphi(400) = 400 - (400/2 + 400/5 - 400/10) = 400 - (200 + 80 - 40) = 160$

$\varphi(160) = 160 - (160/2 + 160/5 - 160/10) = 160 - (80 + 32 - 16) = 64$

Now note that 64 no longer contains a factor of 5. So anything having a common factor with $2^6 = 64$ must have a factor of 2. This is precisely the even numbers, so the odd numbers will be the coprime ones. Thus $\varphi(64) = 32, \varphi(32) = 16$, and so forth, up to $\varphi(2) = 1$. It takes 3 iterations to remove the fives, and then 5 more to get to 2, at which point $\varphi(2) = 1$ and then $\varphi(1) = 1$ and we reach an infinite loop. This totals $3 + 5 = 8$ iterations.

Alternate solution: Upon playing with the function φ, one can realise that it is *multiplicative*, that is, if p and q are themselves coprime, then $\varphi(pq) = \varphi(p)\varphi(q)$. In particular, $\varphi(2^m 5^n) = \varphi(2^m)\varphi(5^n)$. These are easier to compute. In general, any non-trivial factor of p^m for p prime must be a multiple of p, and there are $p^m/p = p^{m-1}$ of these. Thus $\varphi(p^m) = p^m - p^{m-1} = p^{m-1}(p-1)$. In particular, this means that $\varphi(2^m) = 2^{m-1}(2-1) = 2^{m-1}$ and $\varphi(5^n) = 5^{n-1}(5-1) = 5^{n-1}2^2$. Thus $\varphi(2^m 5^n) = 2^{m-1}5^{n-1}2^2 = 2^{m+1}5^{n-1}$. We may iterate this n times before we run out of copies of 5, at which point we will have the number 2^{m+n}, we may iterate $m + n - 1$ more times to get down to 2, at which point $\varphi(2) = 1$. So overall this takes $m + 2n - 1$ iterations. In the case of 1000, we have $m = n = 3$, so $m + 2n - 1 = 3 + 2(3) - 1 = 8$.

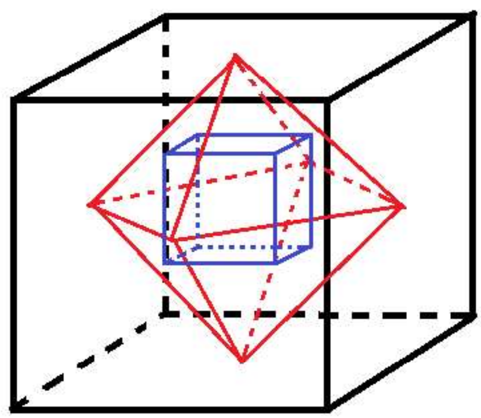
Student Name: ___________________________

Please write your name on ***every*** page.

Answer to F7: 8

F8

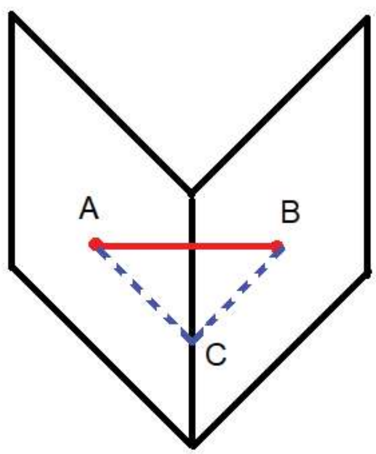
A regular octahedron is inscribed inside a cube, with each vertex of the octahedron being the centre of one of the cube's faces. A smaller cube is inscribed inside the octahedron, with each vertex of the smaller cube being the centre of one of the octahedron's faces. How many times larger is the volume of the large cube compared to the volume of the small cube?



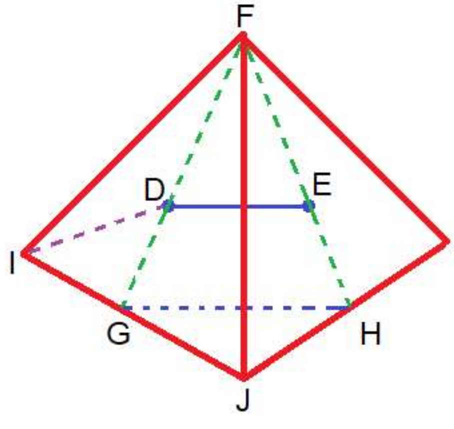
Solution. In order to find the volume of a cube, all we need is the side length. Also, without loss of generality, we can take the side length of the larger cube to be 1 since we are looking for a ratio, to make calculations simpler. First we find the side length of the inscribed octahedron.

Student Name: ______________________________

Please write your name on ***every*** page.



Since A is at the centre of its face and AC forms a right angle with the cube's edge passing through C, $AC = 1/2$. Similarly, $BC = 1/2$. And AC and BC are perpendicular since they are each parallel to edges of the cube that are perpendicular. Thus ABC is right angled at C, and so $AB = \sqrt{(1/2)^2 + (1/2)^2} = 1/\sqrt{2}$. Next we find the side length of the smaller cube.



Student Name: ______________________

Please write your name on ***every*** page.

Clearly triangle GHJ is right angled at J, and we know that $JH = GJ = IJ/2 = 1/2\sqrt{2}$. Thus $GH = \sqrt{(1/2\sqrt{2})^2 + (1/2\sqrt{2})^2} = 1/2$. Since D is the centre of equilateral triangle FIJ, $DI = FD$. Then since $\angle IDF = 120^\circ$, we have that $\angle DIF = \angle IFD = 30^\circ$. Since $\angle GIF = 60^\circ$, we hence have that $\angle GID = 30^\circ$. Since $\angle IGD$ is a right angle, we get that the tringles GID and GIF are similar, and so $FG/FI = GI/DI$. Rearranging, $DI = FI \cdot GI/FG$. But $FI = 1/\sqrt{2}$ and $GI = FI/2 = 1/2\sqrt{2}$. Also, $FG = \sqrt{(1/\sqrt{2})^2 - (1/2\sqrt{2})^2} = \sqrt{3}/2\sqrt{2}$. Thus $FD = DI = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} \cdot \frac{2\sqrt{2}}{\sqrt{3}} = 1/\sqrt{6}$. Noting that triangles FDE and FGH lie in the same plane and thus are similar, we have that $DE/GH = FD/FG$ and so $DE = FD \cdot GH/FG = 1/\sqrt{6} \cdot 1/2 \cdot 2\sqrt{2}/\sqrt{3} = 1/3$. Somehow this inner cube's side length has a integer ratio with the larger cube's side length. As a fun puzzle, try to figure out why!

Since the larger cube's side length is 3 times larger, its volume is $3^3 = 27$ times larger.

Answer to F8: 27