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4 Section D

D1
How many prime numbers less than 100 contain a 1?

Solution. The numbers less than 100 containing a 1 are 1, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 31, 41, 51, 61, 71, 81, 91.
We may immediately eliminate any even numbers since these have a factor of two, as well as 1 itself since it is not prime, so we are left with 11, 13, 15, 17, 19, 21, 31, 41, 51, 61, 71, 81, 91.
Next we eliminate any numbers with digits summing to a multiple of three, since these will be divisible by 3. We are left with 11, 13, 17, 19, 31, 41, 61, 71, 91.
It is clear that none of these are divisible by 5 since they do not end in a 5 or 0. Checking for divisibility by 7, we only eliminate $91 = 7 \times 13$. We are now done, since each of these numbers is less than 100, and so if $a < b$ with $ab = 100$, then $a < \sqrt{100} = 10$. Counting our list, we have 8 such primes.

Answer to D1: 8

D2
What is the probability that a randomly chosen letter of the alphabet is a vowel? (here Y counts as a vowel).

Solution. The vowels, counting Y, are A, E, I, O, U, Y. This is 6 total. There are 26 letters in the alphabet. So the probability is $6/26 = 3/13$.

Answer to D2: 3/13

D3
The songs on an album and their lengths are given in the table below. Which song number is closest to the mean length of a song on the album?

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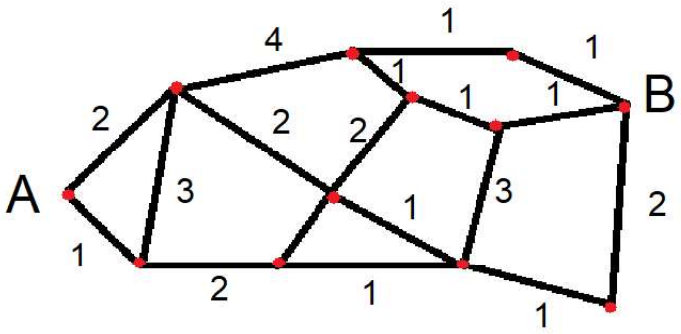
Song	Length (mins:secs)
Song 1	11:26
Song 2	9:47
Song 3	5:00
Song 4	9:00
Song 5	7:58
Song 6	9:13
Song 7	10:02

Solution. We first convert the song lengths to seconds to get $686, 587, 300, 540, 478, 553, 602$. To get the mean, we add these seven numbers and divide by 7 to get approximately 535.14. Of these, 540 is clearly the closest, which is the length of Song 4.

Answer to D3: 4

D4

In the diagram below, what is the length of the shortest path between A and B?



Solution. Through trial and error, we find that the path running along the bottom is the shortest one, and it has length $1 + 2 + 1 + 1 + 2 = 7$.

Answer to D4: 7

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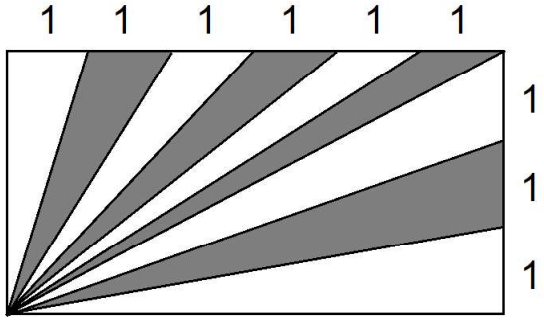
D5

Find the area of the shaded regions in the diagram below:



Solution. Each of these four triangles has a base of length 1. Three of them have a height of 3, while the fourth has a height of 6. Then the total area is $3 \times \frac{1}{2}(1)(3) + \frac{1}{2}(1)(6) = \frac{9}{2} + \frac{6}{2} = \frac{15}{2}$.

Answer to D5: 15/2

D6

What is the smallest positive integer whose digits multiply to 36?

Solution. The factor pairs that multiply to 36 are $(1, 36), (2, 18), (3, 12), (4, 9), (6, 6)$. Since 36 is the product of two digits, it could only be 6×6 or 4×9 here. The smallest of these will occur when we minimize the tens digit, so place 4 there to get the number 49.

Answer to D6: 49

D7

Emily and Logan each pick a number at random from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (they may each pick the same number or different numbers). What is the probability that Emily's number is larger than Logan's number?

Solution. Either someone's number is greater, or they pick the same number. Suppose Emily picks any number. Then the probability of Logan picking the same number is $1/9$, since there are 9 numbers to choose from. Then the probability of someone's number being greater is $1 - 1/9 = 8/9$. In such a scenario, both Emily and Logan have an equal chance of their number being greater, so this is $1/2 \times 8/9 = 4/9$.

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Answer to D7: 4/9

D8

Eugene is a wizard who is in a competition to see who has the strongest spells. Each wizard casts a spell at a watermelon in an attempt to make it explode. To make things more difficult, some glass panels are placed in front of the watermelon. Each glass panel has a 5% chance to block the spell and protect the watermelon. Without any glass panels, Eugene has a 90% chance to destroy the watermelon. With a single glass panel, he has a 5% chance for the glass panel to block the spell, hence a 95% chance for it not to block the spell, and a 90% chance of destroying the watermelon if the spell gets past the glass, for a total of 95% of 90%, which is an 85.5% chance of destroying the watermelon. What is the largest number of glass panels that can be placed in front of the watermelon while still giving Eugene at least a 50% chance of destroying the watermelon?

Solution. Suppose there are n glass panels. The only scenario in which Eugene destroys the watermelon is if panel 1 fails to block his spell AND panel 2 fails to block his spell AND $\cdots$ AND panel n fails to block his spell AND the spell actually destroys the watermelon. In probability, ”AND” means multiplication, so since the probability of any given panel failing to block the spell is $1 - 0.05 = 0.95$ and the probability of the spell destroying the watermelon is $90\% = 0.90$, the total probability is 0.90×0.95^n. We must find the largest n so that $0.90 \times 0.95^n > 0.50$. Using a calculator, we find that $n = 11$ gives approximately 0.512, while $n = 12$ gives approximately 0.486. Thus 11 is the desired maximum number of panels.

Answer to D8: 11