Student Name: ____________________

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4 Section D

D1

What is the units digit in 7^{2019} when expanded?

Solution. We look for a pattern in the units digit of powers of 7. We have $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$, and $7^5 = 16807$. These units digits continue to repeat $7, 9, 3, 1, 7, 9, 3, 1, \ldots$ in a cycle of length 4. At 2019 we will be at the third number in the cycle, so the units digit is 3.

Answer to D1: 3

D2

A square is inscribed inside a circle such that its four vertices touch the circle's circumference. If the circle has an area of 128π cm^2, what is the perimeter of the square?

Solution. The diagonal of the square is a diameter of the circle. The area of a circle with radius r is πr^2, and so if $128\pi = \pi r^2$ then $r^2 = 128$ and so $r = 8\sqrt{2}$. Then $d = 2r = 16\sqrt{2}$. By Pythagorean Theorem we then get that the side length of the square is 16, and so the perimeter of the square is $4 \times 16 = 64$.

Answer to D2: 64

D3

The mean of five numbers is 6. You want to increase the highest and lowest numbers by the same amount in order to make the numbers have a mean of 8. What should you increase them by?

Solution. If five numbers have a mean of 6, then they have a sum of 30. In order to make them have a mean of 8, they need to have a sum of 40, and so we need to add 10 total. We are adding equally to the lowest and highest numbers, so each needs to have $10/2 = 5$ added to it.

Answer to D3: 5

D4
What is the smallest prime number p such that $16p + 1$ is also prime?

Solution. This can be done by trial and error, working our way up from the smallest primes. For $p = 2, 16(2) + 1 = 33$ which is divisible by 3. For $p = 3, 16(3) + 1 = 49$ which is divisible by 7. For $p = 5, 16(5) + 1 = 81$ which is divisible by 3. For $p = 7, 16(7) + 1 = 113$. Testing for all possible divisors up to $\sqrt{113} < 11$ will reveal that this number is indeed prime.

Answer to D4: 7

D5
The year 2019 has 365 days, and January 1st, 2019 is on a Tuesday. How many Fridays are there in 2019?

Solution. If January 1st is on a Tuesday then the first Friday will occur on the 4th day of the year. We need to find the quotient and remainder when 365 is divided by 7. It turns out that $365 = 7 \times 52 + 1$, so the quotient is 52 and the remainder is 1. Then since $7(0) + 4$ is the first Friday and $7(52) + 4$ just exceeds the amount of days in a year, we get that 0 to 51 are all valid Fridays, and hence there are $51 - 0 + 1 = 52$ Fridays in the year.

Answer to D5: 52

D6
If we define the operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d}$, with $\frac{a}{b}$ and $\frac{c}{d}$ fractions in lowest terms, then what is the difference between $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ and $\frac{1}{1} \oplus \frac{1}{2} \oplus \frac{1}{3} \oplus \frac{1}{4} \oplus \frac{1}{5}$?

Solution. We have that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{60}{60} + \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} = \frac{137}{60}$. On the other hand, $\frac{1}{1} \oplus \frac{1}{2} \oplus \frac{1}{3} \oplus \frac{1}{4} \oplus \frac{1}{5} = \frac{2}{3} \oplus \frac{1}{3} \oplus \frac{1}{4} \oplus \frac{1}{5} = \frac{5}{4} \oplus \frac{1}{4} \oplus \frac{1}{5} = \frac{9}{5} \oplus \frac{1}{5} = \frac{14}{6}$. Then the difference is $|\frac{137}{60} - \frac{140}{60}| = \frac{3}{60} = \frac{1}{20}$.

Answer to D6: $\frac{1}{20}$

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D7

Find the number of five-digit positive integers n that satisfy the following conditions:

- The number n is divisible by 5
- The first and last digits of n are equal
- The sum of the digits of n is divisible by 5

Solution. To be divisible by 5, the last digit of n must be either 5 or 0. But the first and last digits of n must be equal, so n must be of the form $5ABC5$ for some A, B, C. Take any two digit number AB (with $A = 0$ a possibility). Then there exist exactly two digits C such that $A+B+C$ is divisible by 5. Since there are 100 possible (A, B) pairs, there are thus $2 \times 100 = 200$ possible values for n.

Answer to D7: 200

D8

How many 3-digit numbers are there, with the property that the digits are in strictly increasing order and the first digit divides the last? (e.g. 123 but not 222)

Solution. We can split this one up into cases. If the initial number is 1, then the third digit can range from 3 to 9. For each such digit k, there are $k - 2$ possibilities for the second number. This gives us $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ possible numbers with first digit 1. If the initial number is 2, then the third digit can be 4, 6, or 8. This gives us $1+3+5 = 9$ possible numbers with first digit 2. If the initial number is 3, then the third digit can be 6 or 9, giving $2 + 5 = 7$ possible numbers. If the initial number is 4, the third digit can only be 8 and so there are 3 possible numbers. These are the only cases possible, since the lowest proper multiple of 5 is 10. Thus there are $28 + 9 + 7 + 3 = 47$ such numbers in total.

Answer to D8: 47