6 Section F

F1
The volume of a cube is 64. What is the total length of all edges of the cube?

Solution. Since the cube has volume 64, each edge has length $\sqrt[3]{64} = 4$. A cube has 12 edges, so the sum of the edge lengths is $12 \times 4 = 48$.

Answer to F1: 48

F2
What is the maximum possible area of a rectangle with perimeter 12?

Solution. A rectangle has 4 sides, but opposite sides have equal length. So let ℓ and b be the variables denoting the length and the breadth of the rectangle.

We are only considering rectangles with perimeter 12, so we are looking for rectangles with $2(\ell + b) = 12$. This equation simplies to

$$b = 6 - \ell$$

This means that we can forget about b and only worry about ℓ as $b = 6 - \ell$.

Now we want to find the maximum possible area of a rectangle with perimeter 12. If we have a rectangle with length ℓ and breadth $12 - \ell$ then its area is

$$\ell \times (6 - \ell)$$

So we want to find the maximum possible value of $\ell(6 - \ell)$. This is a quadratic, so completing the square is something we should always try. This gives us,

$$\ell(6 - \ell) = 6\ell - \ell^2 = 2 \cdot 3 \cdot \ell - \ell^2 = 9 - 9 + 2 \cdot 3 \cdot \ell - \ell^2 = 9 - (\ell - 3)^2$$

So the area of the rectangle with sides ℓ and $6 - \ell$ is actually just $9 - (\ell - 3)^2$ in disguise. We know that $(\ell - 3)^2 \geq 0$ so $-(\ell - 3)^2 \leq 0$ and so

$$9 - (\ell - 3)^2 \leq 9$$

So the area of any such rectangle can be at most 9. Is there some ℓ for which the area is 9? By inspection we see that if $\ell = 3$, then $(\ell - 3)^2 = 0$ and so for such an ℓ, the area is 9.

Thus, 9 is the maximum possible area of a rectangle with perimeter 12.

Answer to F2: 9

F3
A fair 6-sided die is thrown 2018 times. Which of the following is the appoximate probability that it will land on 6 at least 328 times: 0.1, 0.4, 0.7, or 1?

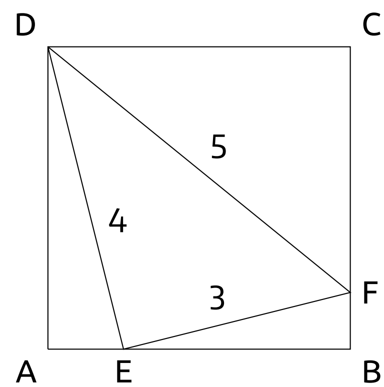
Solution. Over 2018 rolls, a fair die should land on 6 an average of $\frac{2018}{6} = 336\frac{1}{3}$ times. So, we should expect that the probability it lands on 6 at least 328 times is larger than 0.5 but less than 1. So we choose 0.7. In fact, the actual probability is about 0.6994. (Using a computer.)

Answer to F3: 0.7

F4
What is the area of the smallest square that completely covers a triangle with side lengths 3, 4, 5?

Solution.

The first step is to draw a diagram:



Here, $\triangle DEF$ is a 3-4-5 triangle and $ABCD$ is the smallest square that covers it. Since the lengths 3, 4, 5 satisfy the Pythagorean theorem, we know that $\angle DEF = 90^\circ$. Let $\theta = \angle DEA$. Then $\angle FEB = 90^\circ - \theta$. Also, $\angle ADE = 90^\circ - \theta$ and $\angle EFB = \theta$. Hence, $\triangle AED \sim \triangle BEF$. Let $|AE| = x$ and $|AD| = a$. Note that a^2 is the area of the square, which is what we're trying to find. By similar triangles, $\frac{|AD|}{|DE|} = \frac{|BF|}{|FE|} \implies \frac{a}{4} = \frac{a-x}{3}$. Rearranging, we get $a = 4x$. Next, using the Pythagorean theorem on $\triangle AED$, we get $x^2 + a^2 = 16$. Substituting $a = 4x$ into this, we get $a^2 = \frac{256}{17}$.

Answer to F4: $256/17$

F5
Mr. Liang's grade 10 class has 12 boys and 16 girls. For a class project, he randomly partners the students into groups of two. On average, how many groups will have one boy and one girl?

Solution. Consider one group. The probability that the group has one boy and one girl is $\frac{12}{28} \cdot \frac{16}{27} + \frac{16}{28} \cdot \frac{12}{27} = \frac{32}{63}$. ($\frac{12}{28} \cdot \frac{16}{28}$ is the probability that the first person is a boy and the second person is a girl, and $\frac{16}{28} \cdot \frac{12}{27}$ is the probability that the first person is a girl and the second person is a boy.) Hence, the expected number of groups with exactly one boy and one girl is $\frac{32}{63} \times 14 = \frac{64}{9}$.

Answer to F5: $96/7$

F6
A **palindrome** is a number that reads the same forwards and backwards. For example, 12321 is a palindrome, as is 70207 or 8888888. Find any two different whole numbers whose cubes ($n^3 = n \times n \times n$ are 7-digit palindromes. Write both numbers, seperated by a comma, in the blank.

Solution. We know we need a 3-digit number. Identifying symmetrical examples would be ideal. Let $n = 100a + 10b + c$. Then $n^3 = 1000000a^3 + 300000a^2b + 30000a^2c + 30000ab^2 + 6000abc + 1000b^3 + 300ac^2 + 300b^2c + 30bc^2 + c^3$ after expanding. We can see immediately that taking $a = b = c = 1$ will work, resulting in $111^3 = 1367631$. Furthermore, taking $a = c = 1$ and $b = 0$ will work, resulting in $101^3 = 1030301$.

In fact, using a computer we could check that these are the only solutions, but the question does not ask to show they are unique.

Answer to F6: $101, 111$

F7
Let n coins weighing $1, 2, ..., n$ grams be given. Suppose Baron Munchhausen knows which coin weighs how much, but his audience does not. Then $a(n)$ is the minimum number of weighings the Baron must conduct on a balance scale, so as to unequivocally demonstrate the weight of at least one of the coins.

Values of $a(n)$ for small n are as follows:

- $a(1) = 0$ as the audience already knows the only coin has weight 1
- $a(2) = 1$; the Baron may weigh $1 < 2$, identifying both
- $a(3) = 1$; the Baron may weigh $1 + 2 = 3$, which identifies the 3-gram coin
- $a(4) = 1$; the Baron may weigh $1 + 2 < 4$, which identifies the 3-gram coin
- $a(5) = 2$

What is $a(8)$?

Solution. The key to solving this problem is to understand how we can identify a particular coin. A good example is $a(4) = 1$. Here the question says that the Baron may weight $1 + 2 < 4$ which identifies the 3-gram coin. Why is this?

This is because we can never have any other balance with 2 coins on one side and 1 coin on the other such that the scale tips to the side of the side of the side with 1 coin. This only works because $1+2$ is the smallest possible weight of 2 coins and 4 is the smallest possible weight of a single coin when the possible coins are $1, 2, 3, 4$ and we don't have $1+3<4$.

In general, to identify a coin in 1 weighing if it is possible, we want to try to compare a weight of the form $1+2+3+\ldots+k$ with a weight of the form $k+2+\ell+1+\ldots+n$ and use this to identify $k+1$. If we manage to get $1+2+3+\ldots+k=(k+2)+\ldots+n$ for some k, then this automatically identifies $k+1$ since any other sum of k numbers in $1, \ldots, n$ is strictly bigger than $1+2+\ldots+k$ and any other sum of $n-k-1$ numbers is strictly smaller than $k+2+\ldots+n$.

In our question we want to compute $a(8)$. If we are done in 1 weighing then we know that we must have $a(8)=1$ so let's try to look for such a possibility. We want to see if we can identify $k+1$ as above. So let's start with $k=6$. Then we are comparing $1+2+\ldots+6$ with 8. We clearlyhave $1+2+\ldots+6>8$ and there are other solutions to this like $1+2+\ldots+5+7>8$ so this doesn't identify 7.

Never mind. We next try to compare $1+2+\ldots+5=15$ with $7+8=15$. Here we do have

$$1+2+3+4+5=15=7+8$$

and so this automatically identifies 6!

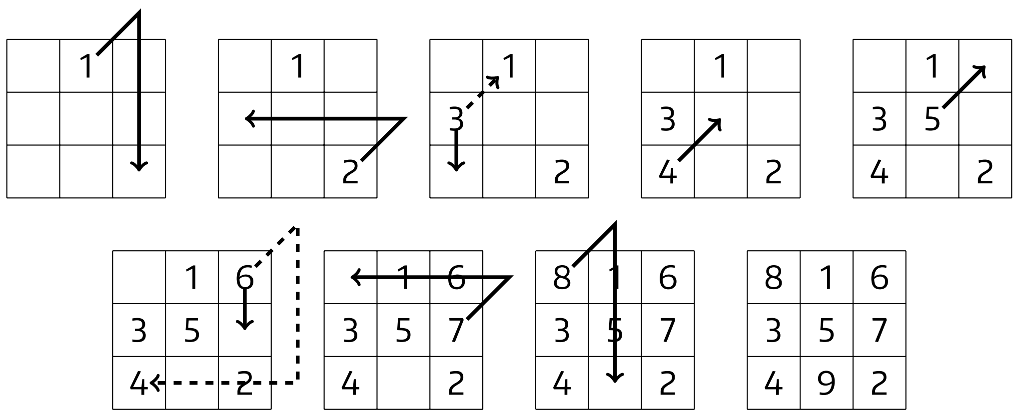
Thus, Baron Munchausen can identify one coin among $1, 2, \ldots, 8$ in just 1 weighing.

As an aside, suppose we had 128915 coins how many weighings would the Baron need? Tanya Khovanova and Konstantin Knop showed that no matter how big n *is,* $a(n)$ *is at most* 2*. Thus, Baron Munchausen can identify 1 particular coin out of* $1, 2, \ldots, 128915$ *coins in just 2 weighings! Their paper on this is on the arxiv and is very readable so take a look if you're interested. https://arxiv.org/pdf/1003.3406.pdf*

Answer to F7: 1

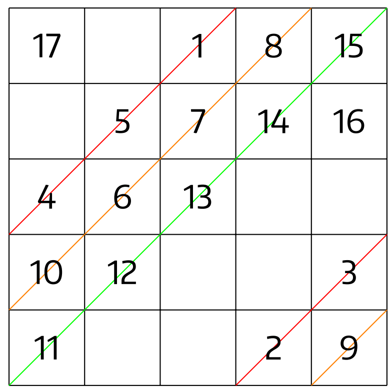
F8

A **magic square** is a grid of whole numbers such that the sum of each row or column is equal to the same number. One way to build a magic squares is to write the numbers in order with the "north-east" technique. Start by writing a 1 in the middle square on the top row. Until the entire square is full, write the next number in the square up and to the right (north-east) of the last number we wrote. If this goes outside the square, wrap around to the other side of the square. Sometimes, that square might already be full; if so, we write the number in the square below the last square (south) instead. See the picture below for an example on a 3×3 magic square:



If you do this process on a 63×63 magic square, what number will be in the top-left corner?

Solution. The crux of this problem is understanding what sort of pattern this "north-east" method actually is describing. It's perhaps useful to try starting it on a 5×5 or 7×7 magic square:



I did not finish labelling this square, because we can already see a pattern. The numbers go north-east (maybe wrapping around) along diagonals (which, because of wraparound, can appear in two places). Once a diagonal is finished, the southward motion starts the next diagonal. In the picture above, the first diagonal is coloured red, the second diagonal orange, and the third diagonal green.

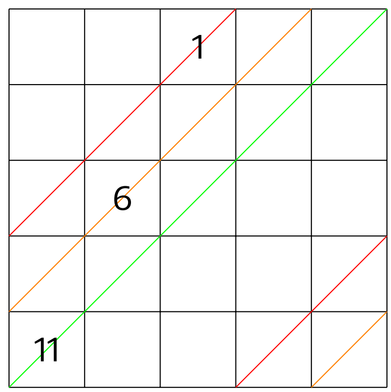
This gives us an idea of how to figure out what number goes into any square: we figure out which diagonal it's on, and then where it is in the diagonal. We care only about the top-left square. We can see that the middle square in the top row is included in the first diagonal (since that is where we start), and each subsequent diagonal includes the next square (to the right) in the top row. This means that, in an $n \times n$ grid, the rightmost square in the top row will be included in the $\frac{n+1}{2}$th diagonal.

The leftmost square in the top row will be in the next diagonal (because of wraparound). So in total, before we even write a number on the same diagonal as the square we want, we will have written

$$\frac{n(n+1)}{2}$$

numbers.

The rest of the problem is figuring out how many additional numbers need to be written after completing $\frac{n+1}{2}$ diagonals. For both the 3×3 and 5×5 cases, we saw that we completed the required number of diagonals and finished on the top-right square. Then, after two more squares (one southward movement and one north-eastern movement, wrapping around) we reach the required top-left squares. Does this pattern continue? In fact, it does. We can see this by looking at the first number we write on each of those diagonals:



Indeed, after exactly $n-1$ north-east movements and 1 southward movement (possibly wrapping around), we always make a movement of two squares south and one square west (possibly wrapping around). Since we started in the middle square on the top, this means we will always start the $\frac{n+1}{2}$th diagonal on the bottom-left square, which means we'll always end it on the top-right square.

Therefore, for an $n \times n$ magic square, we know that the top-left number is written 2 numbers after $\frac{n+1}{2}$ complete diagonals. This means a formula for the number is

$$\frac{n(n+1)}{2} + 2$$

If we substitute $n = 63$, we obtain

$$\frac{63 \times 64}{2} + 2 = \boxed{2018}$$

Answer to F8: 2018