5 Section E

E1

What is the smallest whole number, such that, when you multiply its digits together, you get 80?

Solution. Clearly we can't do it with one digit. Can we do it with 2 digits? We can't, because the largest single-digit factor of 80 is 8, and 88 doesn't work because $8 \times 8 = 64$ is still too small. Are 3 digits enough? Yes! For example, 454 works (because $4 \times 5 \times 4 = 80$). Is 454 the smallest? No! Observe that 258 also works, and is smaller. Convince yourself that $\boxed{258}$ is the smallest number that works.

Answer to E1: 258

E2

An icosahedron is a three-dimensional solid whose faces are regular (equilateral) triangles. An icosahedron has 20 faces. How many edges does an icosahedron have?

Solution. This is a counting problem. There are 20 triangular faces, and each edge must be shared by exactly 2 faces, so there are simply $20 \times 3 \div 2 = \boxed{30}$ edges.

Answer to E2: 30

E3

We define the Levenshtein Distance between two strings to be the minimum number of edits we need to turn one string into another. One edit is either:

- adding a character: CAT $\rightarrow$ CATS
- removing a character: TIME $\rightarrow$ TIE
- turning a character into another: BAKE $\rightarrow$ CAKE

As an example, the Levenshtein Distance between HELLO and MELON is 3. We may, for example, change H to M, then delete one L, and then add an N. There are other ways of achieving the same result, but the best we can do is 3 edits.

What is the Levenshtein Distance between KITTENS and TENTS?

Solution. The easiest way to do this question is by trial-and-error. For example, the following sequence of edits works: KITTENS $\to$ ITTENS $\to$ TTENS $\to$ TENS $\to$ TENTS. Convince yourself that we can't change KITTENS to TENTS using fewer than 4 edits.

Answer to E3: 4

E4

What is the last digit of $1 + 3 + 3^2 + 3^3 + \cdots + 3^{2018}$?

Solution. Notice that we don't care about any digits of each term 3^n except the last one. There's a pattern in those digits: they cycle $1 + 3 + 9 + 7 + 1 + 3 + 9 + 7 + \ldots$. Every group of four digits is divisible by 10 and will have last digit 0, so we're left with just three extra digits $1 + 3 + 9$. Thus the last digit is $\boxed{3}$.

Answer to E4: 3

E5

In Sockland, there is a magical machine that sells socks. When you buy a sock from the machine, it gives you a single sock. The colour of the sock is random, and can be red or blue, with probability 1/2 each. Bobby buys 10 socks from the machine. What is the probability that he ends up with an even number of socks of each colour?

Solution. A useful trick for thinking about problems like this is to *condition on the first 9 socks*. What this means is to imagine that Bobby has 9 socks and is about to buy the 10th sock. Because 9 is an odd number, the first 9 socks have to contain an even number of one colour and an odd number of the other colour. Without loss of generality, we may assume that an even number of the first 9 socks are red and an odd number are blue. Then, Bobby's success entirely depends on the colour of the 10th sock – if it's blue then he ends up with an even number of each colour, and if it's red then he ends up with an odd number of each colour. The probability that the 10th sock is blue is $\frac{1}{2}$. Note that this line of reasoning works *no matter what the colours of the first 9 socks are*. There is always exactly one colour for the 10th sock that makes things work, and the probability of getting that colour for the 10th sock is $\frac{1}{2}$.

Answer to E5: 1/2

E6

Define $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. (This is the Fibonacci sequence, and it starts $1, 1, 2, 3, 5, 8, \ldots$; each number is the sum of the previous two numbers.) Calculate $F_{31} \times F_{31} - F_{30} \times F_{32}$.

Solution. The question asks for $F_{31}^2 - F_{30}F_{32}$, but there's probably nothing special about the number 31. Let's see if we can find a general pattern. The first few terms in the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, ...So,

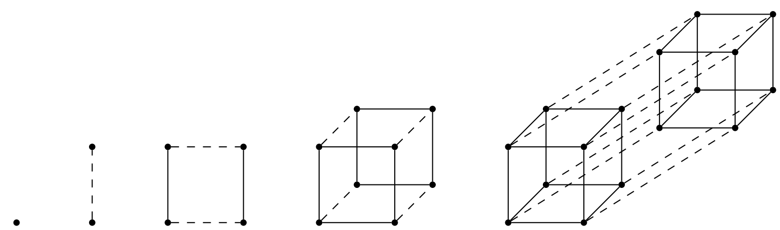
- $F_2^2 - F_1F_3 = 1^2 - (1)(2) = -1$
- $F_3^2 - F_2F_4 = 2^2 - (1)(3) = 1$
- $F_4^2 - F_3F_5 = 3^2 - (2)(5) = -1$

It seems like the numbers alternate between -1 and 1! From this, we guess that $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$, so that $F_{31}^2 - F_{30}F_{32} = 1$. Let's see if we can prove this. (Note: Students were not expected to see the proof on the actual contest, only write out the first few terms and observe the pattern.) Define $G_n = F_n^2 - F_{n-1}F_{n+1}$. Our goal is to show that $G_n = (-1)^{n+1}$. Since we already checked that $G_2 = -1$, it suffices to show that $G_{n+1} = -G_n$ for all $n \geq 2$. This can be shown using a direct calculation:

$$
\begin{aligned}
G_{n+1} &= F_{n+1}^2 - F_nF_{n+2} \\
&= F_{n+1}^2 - F_n(F_n + F_{n+1}) \\
&= F_{n+1}^2 - F_nF_{n+1} - F_n^2 \\
&= F_{n+1}^2 - F_nF_{n+1} - (F_n^2 - F_{n-1}F_{n+1}) - F_{n-1}F_{n+1} \\
&= F_{n+1}^2 - F_nF_{n+1} - G_n - F_{n-1}F_{n+1} \\
&= F_{n+1}^2 - F_{n+1}(F_{n-1} + F_n) - G_n \\
&= F_{n+1}^2 - F_{n+1}^2 - G_n \\
&= -G_n
\end{aligned}
$$

Answer to E6: 1

E7
A 0-hypercube is a point. A 1-hypercube is a line segment, and is formed by connecting the vertices of two 0-hypercubes. A 2-hypercube is a square, and is formed by connecting the corresponding vertices of two 1-hypercubes. Similarly, a 3-hypercube is a cube, formed by connecting the corresponding vertices of two 2-hypercubes, and a 4-hypercube is called a tesserect, formed by connecting the corresponding vertices of two 3-hypercubes. We can keep continuing this pattern. Two dimensional drawings of 0, 1, 2, 3, and 4-hypercubes are provided in this picture:



How many edges does a 6-hypercube have?

Solution. There is a pattern: Every time we extrude into a new dimension, we have two copies of the lower-dimensional object, plus additional edges for each vertex. Another way to say this is that the number of edges in dimension $n + 1$ is equal to two times the number of edges in dimension n plus the number of vertices in dimension n. Thus we can calculate:

- 4 edges in dimension 2
- $4 \times 2 + 4 = 12$ edges in dimension 3
- $12 \times 2 + 8 = 32$ edges in dimension 4
- $32 \times 2 + 16 = 80$ edges in dimension 5
- $80 \times 2 + 32 = 192$ edges in dimension 6

Answer to E7: 192

E8

A **message** is a string of 0s and 1s; each digit in the message is called a **bit**. Rohan wishes to send Jasmine the 4-bit message 0101 across an unreliable communication channel. It is possible for some bits to be flipped from 0 to 1 or 1 to 0. Each bit is flipped with probability $2/5$.

To make it more likely the message is delivered correctly, Rohan sends each bit of the message three times, so the message he transmits is 000 111 000 111.

Jasmine will try to decipher Rohan's message by looking at each 3-bit block, and guess the original bit was:

$$\begin{cases} 0 & \text{if she sees 000, 001, 010, or 100} \\ 1 & \text{if she sees 111, 110, 101, or 011} \end{cases}$$

For instance, Jasmine could receive the message $\underline{1}0\underline{1}$ 111 00$\underline{1}$ 1$\underline{0}$1, where the underlined bits have been flipped from their original values. Jasmine would first look at the first 3 bits, and since she sees 101, would guess 1 as the bit Rohan intended to send. Jasmine continues to guess the whole message as 1101, which is not correct.

What is the probability that Jasmine will guess the correct message exactly? Write the answer in the form a^b/c^d where a and c are prime numbers and b and d are positive exponents.

Solution. Jasmine guesses the correct message if and only if in each block of 3 bits, at most 1 bit is flipped. The probability that at most 1 bit is flipped in a block of 3 bits is

$$\Pr(\text{0 bits are flipped}) + \Pr(\text{1 bit is flipped}) = \left(\frac{3}{5}\right)^3 + 3 \cdot \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^2 = \frac{81}{125} = \frac{3^4}{5^3}.$$

The probability that this happens in all 4 blocks is $\left(\frac{3^4}{5^3}\right)^4 = \frac{3^{16}}{5^{12}}$.

Answer to E8: $3^{16}/5^{12}$