Section E

E1
Ambrose tosses 100 fair coins. What is the probability that an even number of them come up heads?

Solution. As the probability of flipping a head versus flipping a tail is the same for each coin, there are only two possible outcomes:

1. An even number of coins show heads (an even number of coins would show tails to add up to 100 total)
2. An odd number of coins show heads (an odd number of coins would show tails to add up to 100 total)

Thus, the probability that an even number of coins come up heads is $\frac{1}{2}$.

Answer to E1: $1/2$

E2
Alfredo, Cameron, Kasi, Hugo and Shin are to be seated around a circular table. Alfredo refuses to sit next to Cameron. How many possible different seating arrangements are there?

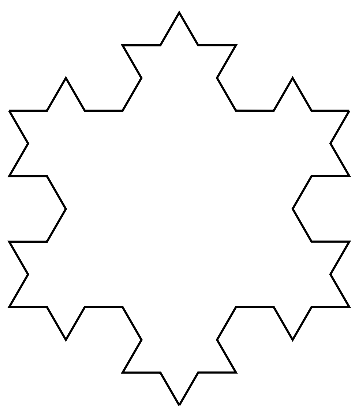
(Note: The table is perfectly circular and does not have a starting point, so ACKHS, CKHSA, KHSAC, HSACK and SACKH are all the same seating arrangment. However, you can tell the difference between being seated clockwise and counterclockwise, so ACKHS and SHKCA are different.)

Solution. We may always represent a seating arrangement by a permutation of ACKHS which begins with A since any choice of starting point gives the same seating arrangement. Furthermore, every distinct permutation starting with A gives a distinct seating arrangement under the rules.

A seating arrangement where Alfredo does not sit next to Cameron corresponds to a permutation of the form **A**X**C**YZ or **A**XY**C**Z, where XYZ is a permutation of KHS. There are two choices for the position of Cameron with respect to Alfredo and $6 = 3!$ permutations of KHS, so there are 12 seating arrangements in total where Alfredo does not sit next to Cameron.

Answer to E2: 12

E3
The area of this figure is $\frac{5}{6}\sqrt{3}$. The perimeter is one of the following: 0.8, 8, 80, or 800. What is it?



Solution. This is just an estimation problem. 0.8 is way too small and 80 is way too big, so 8 must be the answer.

Answer to E3: 8

E4
What is the expected number of times a fair coin needs to be tossed in order to see 2 consecutive heads?

Solution. Let x be the expected number of times a fair coin needs to be tossed in order to see 2 consecutive heads. We consider cases. I'll use H and T to denote heads and tails, respectively.

Case 1. We start with T. Then we have to score 2 consecutive heads in the remaining tosses, except we wasted the first toss. So, we expect $1 + x$ tosses in this case.

Case 2.1 We start with HT. Then we have to score 2 consecutive heads in the remaining tosses, except we wasted the first two tosses. So we expect $2 + x$ tosses.

Case 2.2 We start with HH. Then we're done, and it took us 2 tosses.

Now, Case 1 has a $\frac{1}{2}$ chance of occurring, and Cases 2.1 and 2.2 each have a $\frac{1}{4}$ chance of occurring. This gives us an equation for x:

$$x = \frac{1}{2}(1 + x) + \frac{1}{4}(2 + x) + \frac{1}{4}(2)$$

Solving for x, we get $x = 6$.

Answer to E4: 6

E5

A triangle has side lengths 6, 8, 10. A point P is chosen randomly inside the triangle. What is the probability that the distance from P to the nearest vertex is more than 2?

Solution. This is just an area problem. Take the area of the triangle and subtract the area of the three sectors from each vertex with radius 2. In total the sectors cover an angle of 180 deg because the total sum of angles in a triangle is that. That's half a circle. Thus we get $6 \times 8/2 - 2 \times 2 \times \pi/2 = 24 - 2\pi$. Now we need to divide by the total area, which is 24, to get $1 - \pi/12$.

Answer to E5: $1 - \pi/12$

E6

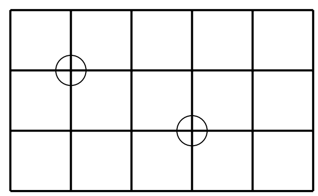
How many ways are there to arrange the numbers 1, 2, 3, 4, 5, 6 in a row such that the sum of any two adjacent numbers is odd?

Solution. In order for the sum of any two adjacent numbers to be odd, the numbers must be arranged in the following patterns (i.e. you cannot have two odd numbers adjacent to each other or two even numbers adjacent to each other): even odd even odd even odd, or odd even odd even odd even.

There are 3 odd numbers and 3 even numbers. Therefore, the total number of possibilities is $3 \times 3 \times 2 \times 2 \times 1 \times 1$ for each of the two patterns, and hence there are $36 + 36 = 72$ total ways.

Answer to E6: 72

E7
If each move may only be a single point up or right and the indicated points may not be crossed, how many routes are there from the bottom left corner to the top right?



Solution. One way to do such problems is to solve similar problems for each point in between the bottom left and the top right, which can be done by adding together the values to the left and bottom. Repeating this process gives us 17 ways.

Answer to E7: 17

E8
What is the smallest n such that 2^{15} is a factor of n!?

Solution. We can check that

$$\begin{aligned} 16! &= 16 \times 14 \times 12 \times 10 \times 8 \times 6 \times 4 \times 2 \text{ mod } 2^{1}5 \\ &= 16 \times 2 \times 4 \times 2 \times 8 \times 2 \times 4 \times 2 \text{ mod } 2^{1}5 \\ &= 2^{4+1+2+1+3+1+2+1} \text{ mod } 2^{1}5 \\ &= 2^{1}5 \text{ mod } 2^{1}5 \\ &= 0 \text{ mod } 2^{1}5 \end{aligned}$$

and clearly, 15! won’t be enough. Thus 16 is the smallest such n.

Answer to E8: 16