Section F

F1
If $-5 \leq x \leq 3$ and $-2 \leq y \leq 7$, find the maximum possible value of $x^2 - y^2$.

Solution. Note that $x^2 - y^2$ is maximized exactly when x^2 is maximized and y^2 is minimized. The maximum value of x^2 for $-5 \leq x \leq 3$ is 25, attained when $x = -5$. The minimum value for y^2 for $-2 \leq y \leq 7$ is 0 (since $y^2 \geq 0$), attained when $y = 0$. This gives a maximum value of 25 for $x^2 + y^2$.

Answer to F1: 25

F2
Find the greatest real value of x that satisfies the nonic equation

$$1 + x + x^2 + x^3 + \cdots + x^9 = 1023$$

Solution. By inspection,

$$1 + 2 + 2^2 + \cdots + 2^9 = 2^{10} - 1 = 1023.$$

Moreover, $x = 2$ is clearly the maximum value which satisfies the given equation, since $1 + x + x^2 + \cdots + x^9$ is strictly increasing for $x > 0$.

Answer to F2: 2

F3
Two people each arrive at a restaurant at a random instant between 5 PM and 6 PM. Each person stays for half an hour. What is the probability that there will be some point in time when both of them are at the restaurant?

Solution. Let x and y be the fractions of an hour past 5 PM at which the two people arrive at the restaurant. Since both people arrive between 5 PM and 6 PM, $0 \leq x, y \leq 1$. Also, there is some time at which both people are at the restaurant at the same time if and only if $|x - y| \leq \frac{1}{2}$. To find the probability that such an event occurs, we divide the area of the region defined by $|x - y| \leq \frac{1}{2}$, $0 \leq x, y \leq 1$ (the set of favourable outcomes), which is the shaded region in Figure 1, by the area of the region defined by $0 \leq x, y \leq 1$ (the event space). The former has area $\frac{3}{4}$, while the latter has area 1, so the probability that there will be a point in time when both people are at the restaurant is $\frac{3}{4}$.

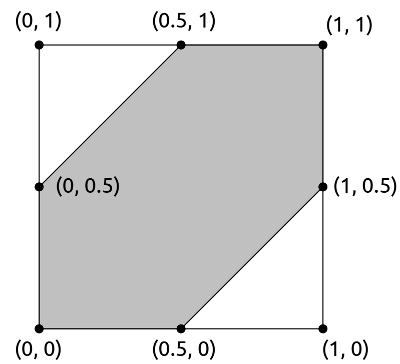


Figure 1: The shaded area represents the cases where the two people arrive within 30 minute of each other

Answer to F3: $\frac{3}{4}$

F4

Find all values of x such that $\log_2(\log_4(x)) = \log_4(\log_2(x))$.

Solution. We have

$$\log_2(\log_4(x)) = \log_4(\log_2(x)) \tag{1}$$
$$\log_2(\log_4(x)) = \frac{1}{2}\log_2(\log_2(x)) \tag{2}$$
$$\log_4(x) = (\log_2(x))^{1/2} \tag{3}$$
$$\frac{1}{2}\log_2(x) = (\log_2(x))^{1/2} \tag{4}$$
$$(\log_2(x))^{1/2} = 2 \qquad \log_2(x) \neq 0 \text{ by (1)} \tag{5}$$
$$\log_2(x) = 4 \tag{6}$$
$$x = 16 \tag{7}$$

Answer to F4: 16

F5

There are 25 poles on a bike rack, each of which is currently holding 2 bikes. The 50 bikers who have attached their bikes to the rack leave in a random order. After 46 people leave, what is the probability that there's still a pole with 2 bikes attached?

Solution. It is easier to find the probability that each of the poles has at most one bike attached. We will calculate the probability by treating each bike as distinguishable, partitioning the fifty bikes into pairs (one for each pole), and counting combinations of these bikes. The number of combinations of poles having one bike attached to them is $\binom{25}{4}$, and for each of those poles there are two choices for the bike which is attached. Since there are $\binom{50}{4}$ ways to choose any four bikes from the fifty, the probability that each of the poles has at most one bike attached is

$$\frac{2^4\binom{25}{4}}{\binom{50}{4}} = \frac{2^4 \cdot 25 \cdot 24 \cdot 23 \cdot 22}{50 \cdot 49 \cdot 48 \cdot 47} = \frac{2024}{2303}.$$

Therefore, the probability that there is a pole with two bikes attached is $1 - \frac{2024}{2303} = \frac{279}{2303}$.

Answer to F5: $\frac{279}{2303}$

F6

Find the smallest positive integer n that satisfies

$$\gcd(n^2 + 4n - 5, 2n^2 + 9n - 5) = 2016$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

Solution. We use the fact that if a, b are integers, then $\gcd(a, b) = \gcd(b, a - qb)$ for any integer q. (*Note. If you have seen the Euclidean Algorithm, this fact is the reason it works.*)

Since $2n^2 + 9n - 5 - 2(n^2 + 4n - 5) = n + 5$, we have

$$\gcd(2n^2 + 9n - 5, n^2 + 4n - 5) = \gcd(n^2 + 4n - 5, n + 5).$$

Since $n^2 + 4n - 5 = n(n + 5) - n - 5$, we have

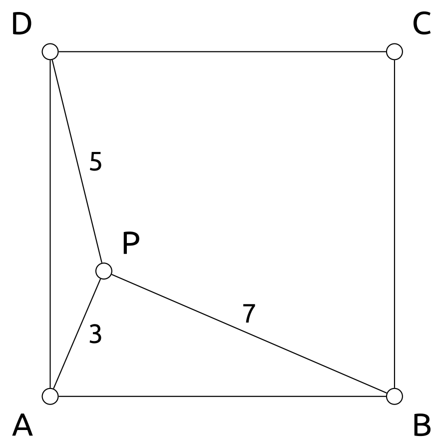
$$\gcd(n^2 + 4n - 5, n + 5) = \gcd(-n - 5, n + 5) = n + 5.$$

We want this to equal 2016, so we must have $n + 5 = 2016$. Therefore $n = 2011$. (In fact, this shows that $n = 2011$ is the *only* positive integer with $\gcd(2n^2 + 9n - 5, n^2 + 4n - 5) = 2011$.)

Answer to F6: 2011

F7

Point P is inside square ABCD with $PA = 3$, $PB = 7$, and $PD = 5$. What is the area of the square ABCD?



Solution. We will use coordinates to solve this problem. Let x denote the side length of the square. Let $A = (0, 0)$, $B = (x, 0)$, $C = (x, x)$, and $D = (0, x)$. Also, let $P = (a, b)$. Then,

$$PA = 3 \iff a^2 + b^2 = 9, \tag{8}$$

$$PB = 7 \iff (a - x)^2 + b^2 = 49 \iff a^2 - 2ax + x^2 + b^2 = 49, \quad \text{and} \tag{9}$$

$$PD = 5 \iff a^2 + (b - x)^2 = 25 \iff a^2 + b^2 - 2bx + x^2 = 25. \tag{10}$$

Substituting $a^2 + b^2 = 9$ into equations (2) and (3), we get the two equations

$$x^2 - 2ax = 40 \tag{11}$$

$$x^2 - 2bx = 16 \tag{12}$$

Rearranging and then squaring both sides, we get

$$2ax = x^2 - 40 \implies 4a^2x^2 = x^4 - 80x^2 + 1600 \tag{13}$$

$$2bx = x^2 - 16 \implies 4b^2x^2 = x^4 - 32x^2 + 256 \tag{14}$$

Adding these two equations together, we get

$$4(a^2 + b^2)x^2 = 2x^4 - 112x^2 + 1856.$$

Using $a^2 + b^2 = 9$ gives

$$36x^2 = 2x^4 - 112x^2 + 1856.$$

Rearranging and simplifying, this becomes

$$x^4 - 74x^2 + 928 = 0.$$

Now use the quadratic formula to solve for x^2. We obtain

$$x^2 = \frac{74 \pm \sqrt{74^2 - 4 \cdot 928}}{2} = 16 \text{ or } 58.$$

Note that $x^2 = 16$ is impossible, since this would imply that the side length of the square is 4. Looking at $\triangle ABP$, this would mean that $\triangle ABP$ has sides of length 3, 4, and 7, which is impossible since these side lengths violate the triangle inequality. Therefore $x^2 = 58$, so the area of ABCD is 58.

Answer to F7: 58

F8

Determine the number of 4-tuples (a, b, c, d) of (not necessarily distinct) positive integers whose product, $abcd$, is 2016. You may use the fact that $2016 = 2^5 \cdot 3^2 \cdot 7$.

Solution. (*Note. This solution is intentionally more verbose than needed. Hopefully this will allow students who have never seen these counting techniques before to understand what is happening.*)

We need to make use of the fact that the prime factorization of 2016 is $2^5 \times 3^2 \times 7$. To find all possible ordered 4-tuples (a, b, c, d) of positive integers with $abcd = 2016$, think of a, b, c, d as *buckets* labelled a, b, c, d, and think of putting the prime factors into these buckets. The problem becomes to find the number of ways to distribute five 2's, two 3's, and one 7 among these four buckets. Each possible distribution of prime factors into buckets corresponds to one solution (a, b, c, d), and each solution (a, b, c, d) corresponds to a distribution of prime factors into buckets.

First, we calculate the number of ways to distribute five 2's into four buckets. This can be done using a technique called "stars and bars". Imagine each 2 as a star, and imagine inserting three dividers between the stars. The number of stars before the first divider is the number of 2's bucket a gets, the number of stars between the first and second dividers is the number of 2's bucket b gets, and so on, until the number of stars after the last divider is the number of 2's bucket d gets.

$$\underbrace{\star}_{a} \mid \underbrace{\star\star}_{b} \mid \underbrace{\star}_{c} \mid \underbrace{\star}_{d}$$

For example, for the stars and bars arrangement above, a gets one star, b gets two stars, c gets one star, and d gets one star. It is possible for two dividers to go into the same slot. For example:

$$\star \mid\mid \star\star\star\star \mid$$

This means that a gets 1 star, b gets *no* stars, c gets 4 stars, and d gets *no* stars. Convince yourself that every arrangement of 5 stars and 3 bars corresponds to a distribution of five 2's among four buckets, and vice-versa. Thus the number of ways to distribute five 2's among four buckets is equal to the number of ways to arrange 5 stars and 3 bars.

Now, the number of ways to arrange 5 stars and 3 bars is precisely $\binom{8}{3}$. This is because every arrangement of these 8 items can be obtained by first picking one of the $\binom{8}{3}$ ways to put 3 bars into 8 spots, and then putting the 5 stars in the remaining 5 spots. Thus there are $\binom{8}{3}$ ways to distribute the five factors of 2 among the four buckets a, b, c, d.

Repeating the same reasoning as above, the number of ways to distribute two 3's among four buckets is $\binom{5}{3}$, and the number of ways to distribute one 7 among four buckets is $\binom{4}{3}$.

Therefore the total number of ways to distribute all the factors among a, b, c and d is

$$\binom{8}{3}\binom{5}{3}\binom{4}{3} = \frac{8 \times 7 \times 6}{3!} \cdot \frac{5 \times 4 \times 3}{3!} \cdot \frac{4 \times 3 \times 2}{3!} = 2240.$$

Answer to F8: 2240