Section E

E1

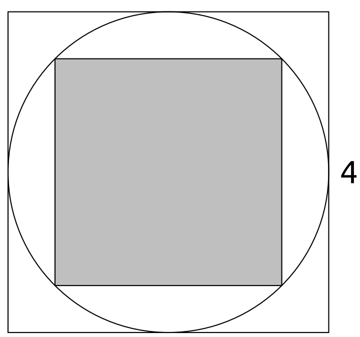
Alice tosses 3 coins and Bob tosses 1 coin. What is the probability that Alice gets more heads than Bob?

Solution. If Bob tosses a head, then Alice has $\frac{1}{2}$ chance of getting at least two heads. If Bob tosses a tail, then Alice has $\frac{7}{8}$ chance of getting at least one head. So the total probability is $\frac{11}{16}$.

Answer to E1: $\frac{11}{16}$

E2

Inscribe a square in a circle, and that circle in a square. If the outer square has side length 4, what is the area of the inner square?



Solution. The inner square is a kite with diagonals of length 4. So its area is

$$\frac{1}{2} \times 4 \times 4 = 8$$

Answer to E2: 8

E3

How many six-digit palindromes are divisible by 11?

Note: A palindrome is a number that reads the same forwards and backwards. For example, 123321 is a palindrome, as is 702207 or 888888. Do not include numbers like 012210 — that is a five-digit number, not a six-digit number.

Solution. All six-digit palindromes are divisible by 11. It is easy to see that there are 900 of them.

Answer to E3: 900

E4
What is the remainder when

$$2015^{2016^{2017}}$$

is divided by 9? Note that in the standard order of operations, the exponent is evaluated before the base, so for example $2^{1^2} = 2^1 = 2$.

Solution.

$$2015^{2016^{2017}} \equiv (-1)^{2016^{2017}} \equiv 1^{1008 \cdot 2016^{2016}} \equiv 1 \pmod 9$$

Answer to E4: 1

E5
What is the *least* possible number of elements that must be deleted from the set

$$\{1, 2, 3, \ldots, 20\}$$

so that the product of the remaining numbers is a perfect square?

Solution. We must remove 19, 17, 13, 11. It turns out that

$$1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times12\times14\times15\times16\times18\times20 = 52672757760000 = 7257600^2$$

is a perfect square, so we're done.

Answer to E5: 4

E6
A car travels downhill at 120 km/h, on flat roads at 96 km/h, and uphill at 80 km/h. The car takes 4 hours to travel from town A to town B, and 5 hours for the return trip. Find the distance between the two towns in kilometres.

Solution. On the way from town A to town B, the car spends a time of $a/120$ driving downhill, $b/96$ driving on flat roads, and $c/80$ driving uphill. Since the total driving time from town A to town B is 4 hours, this gives us the equation

$$\frac{a}{120} + \frac{b}{96} + \frac{c}{80} = 4.$$

On the trip back, roads that were downhill when we drove from A to B now become uphill from B to A, and roads that were uphill become downhill (flat roads are still flat). This gives us the equation

$$\frac{a}{80} + \frac{b}{96} + \frac{c}{120} = 5.$$

Adding these two equations gives

$$\frac{a}{120} + \frac{b}{96} + \frac{c}{80} + \frac{a}{80} + \frac{b}{96} + \frac{c}{120} = 9.$$

Combining the fractions into a common denominator and simplifying, we get

$$\frac{a+b+c}{48} = 9,$$

so the distance between the two towns is $a + b + c = 48 \times 9 = 432$.

Answer to E6: 432

E7

How many positive integers smaller than 729 are relatively prime to 729? You may use the fact that $729 = 3^6$.

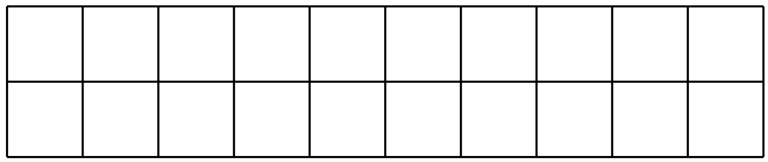
Note: Two numbers are relatively prime if their greatest common divisor is 1; that is, their only positive common factor is 1.

Solution. Note that since $729 = 3^6$, a positive integer n is relatively prime with 729 if and only if n is not a multiple of 3. Between 1 and 729, there are $729/3 = 243$ multiples of 3. All the other numbers are not multiples of 3. Thus, there are $729 - 243 = 486$ numbers between 1 and 729 that are not multiples of 3 (and hence are relatively prime to 729).

Answer to E7: 486

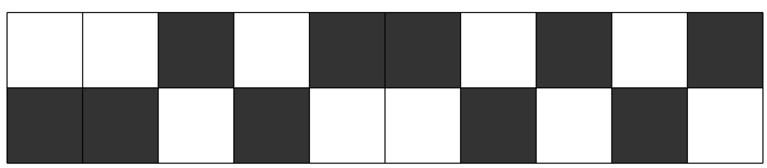
E8

How many ways are there to colour some cells of the below 2×10 grid black, such that each 2×2 square has exactly two black grid cells?



Solution. Suppose we colour no squares in the first column. Then the only colouring that works is this one:

Finally, suppose we colour exactly one square in the first column. For there to be exactly two black squares in each 2×2 square, we must colour *exactly* one square from each of the other columns as well. (Convince yourself of this.)



For each column, we have two choices for which square to colour. Since there are 10 columns, this gives a total of $2^{10} = 1024$ colourings.

Thus, the total number of colourings is $1 + 1 + 1024 = 1026$.

Answer to E8: 1026