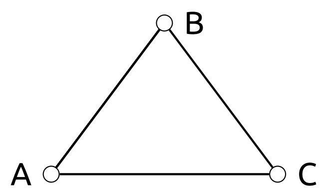
Section D

D1
Isosceles triangle ABC has $AB = BC \neq CA$, and its base CA is its longest side. All sides have lengths that are positive integers. If its perimeter is 16 units, what is its height?

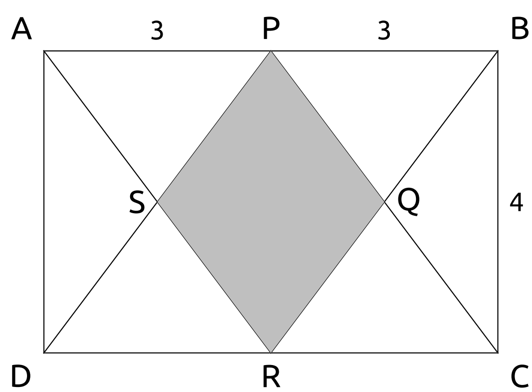
Note: An isosceles triangle is a triangle with at least two sides of equal length.



Solution. The only possibility is $AB = BC = 5$ and $CA = 6$. By the Pythagorean Theorem, the resulting height is 4.

Answer to D1: 4

D2
In rectangle ABCD, $AB = 6$ and $BC = 4$. P is the midpoint of AB and R is the midpoint of CD. If PC and BR intersect at Q and AR and PD intersect at S, what is the area of PQRS?



Solution. This is a kite with diagonals of length 3 and 4. The area is

$$A = \frac{1}{2}D_1D_2 = \frac{1}{2} \cdot 3 \cdot 4 = 6$$

Answer to D2: 6

D3
During a sale, the price of a book is discounted 20%. By how much (as a percentage of the new price) must that be increased to return the book to its usual price?

Solution. Since
$$\frac{1}{0.8} = 1.25$$
therefore the price must be increased by 25%.

Answer to D3: 25%

D4
There are 48 tourists in a group. All of them can speak at least one of Spanish and French. 26 of them can speak Spanish and 31 can speak French. How many tourists can speak both Spanish and French?

Solution. We compute
$$26 + 31 - 48 = 9$$
people that can speak both languages.

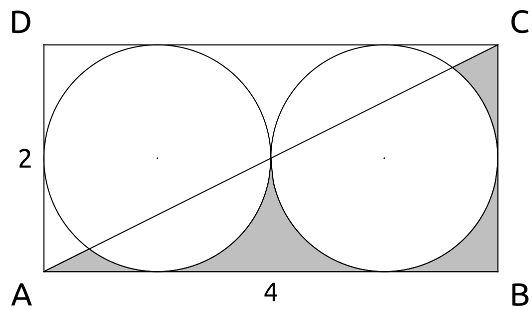
Answer to D4: 9

D5
The side length of a blue cube is $\frac{3}{4}$ the side length of a red cube. What is the ratio of the volume of the blue cube to the volume of the red cube?

Solution. We compute
$$\left(\frac{3}{4}\right)^3 = \frac{27}{64}$$
which is the ratio of volumes.

Answer to D5: $\frac{27}{64}$

D6
Rectangle ABCD has length 4 and width 2. Two circles of radius 1 are drawn inside the rectangle as shown below. Find the area of the shaded region.



Solution. The rectangle has area $2 \times 4 = 8$. The circles each have area π. Collectively, both circles have area 2π. The rectangle, minus the circles, has area $8 - 2\pi$. Half of this region is shaded, and so the shaded region has area $4 - \pi$.

Answer to D6: $4 - \pi$

D7
What is the area of an equilateral triangle with height 3? Write your answer in the form $a\sqrt{b}$ where a and b are positive integers, and b is as small as possible.

Solution. We simply use the Pythagorean Theorem to obtain a base of $2\sqrt{3}$, and hence an area of $3\sqrt{3}$.

Answer to D7: $3\sqrt{3}$

D8
A calendar date is called *productive* if the product of its day and month is equal to the last two digits of its year. For example, 2016-04-04 (April 4, 2016) is productive since $4 \times 4 = 16$. How many dates between January 1, 2000 and December 31, 2999 are productive?

Solution. (*Note.* We do not need to consider leap years, because for the date February 29th, the product of its day and month is $2 \times 29 = 58$, which cannot be the last two digits of leap year because leap years are always divisible by 4.)

The total number of month/day pairs listed in the table is

$$31 + 28 + 31 + 24 + 19 + 16 + 14 + 12 + 11 + 9 + 9 + 8 = 212$$

Finally, observe that each entry in the table gives us **10** possible productive dates between January 1, 2000 and December 31, 2999, because for each month/day pair there are 10 years that work. For example, for the month/day pair March 15, the corresponding 10 productive dates are 2045-03-15, 2145-03-15, 2245-03-15, ..., and 2945-03-15. Therefore, there are $212 \times 10 = 2120$ productive dates between January 1, 2000 and December 31, 2999.

Answer to D8: 2120