Part F

1. Let n be the mystery number. Since 3 is the remainder when n is divided by 4, the quotient when n is divided by 4 is $\frac{n-3}{4}$. Similarly, the quotients when n is divided by 10 and 12 are $\frac{n-1}{10}$ and $\frac{n-3}{12}$, respectively.

   Using clue (d), we get the equation

$$\frac{n-3}{4}+\frac{n-1}{10}+\frac{n-3}{12}=16+\frac{n}{3}$$

   which solves to give $n=\boxed{171}$.

2. We use the following algebra trick. For any positive number k, we have

$$\frac{1}{\sqrt{k}+\sqrt{k+1}}=\frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k+1}-\sqrt{k})}=\sqrt{k+1}-\sqrt{k}.$$

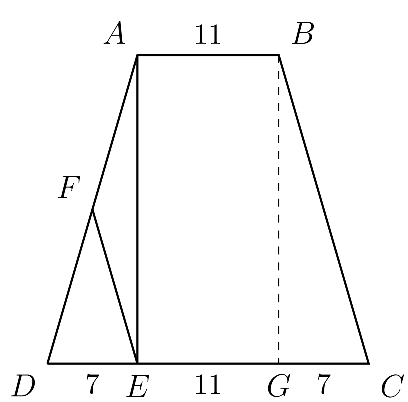
   (Here, we multiplied the numerator and denominator by $\sqrt{k+1}-\sqrt{k}$ so that the denominator simplifies to 1 by difference of squares.)

   Thus,

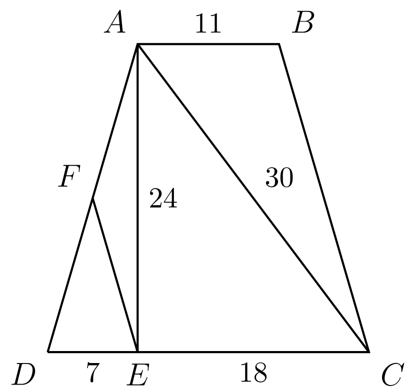
$$\begin{aligned}&\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{n-1}+\sqrt{n}}\\&=\left(\sqrt{2}-1\right)+\left(\sqrt{3}-\sqrt{2}\right)+\cdots+\left(\sqrt{n}-\sqrt{n-1}\right).\\&=\sqrt{n}-1 \quad \text{[the middle terms cancel]}\end{aligned}$$

   So the equation becomes $\sqrt{n}-1=4$, which solves to give $n=\boxed{25}$.

3. Drop a perpendicular from B to DC, and call the foot G. Then $EG=AB=11$. Then, $DE+CG=14$. Since $DE=CG$ by symmetry, $DE=CG=7$.

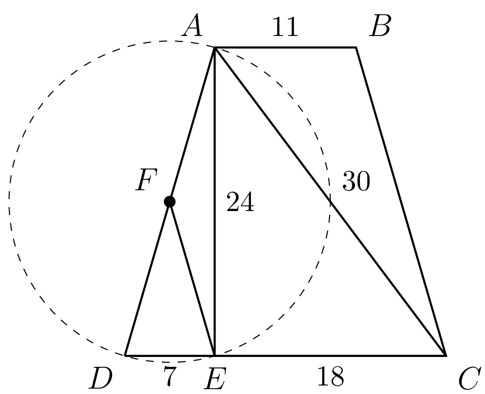


   Next, we have $EC=EG+GC=11+7=18$. By the Pythagorean Theorem on ΔAEC, $AE=\sqrt{30^2-18^2}=24$.



By the Pythagorean Theorem on ΔAED, $AD = \sqrt{24^2 + 7^2} = 25$.

Since ΔAED is a right triangle and F is the midpoint of its hypotenuse, F must be the centre of its circumcircle:



The radius of the circle is $r = \frac{1}{2} \cdot AD = \frac{25}{2}$.

Since FE is a radius of this circle, $FE = \boxed{\frac{25}{2}}$.

4. Given any colouring, there are 4 possible ways to rotate the board. (You can rotate it by 90 degrees clockwise 0, 1, 2, or 3 times. Of course, it might not be true that every rotation produces a new colouring.)

   If you colour two squares on the board, rotating it once will never produce the same colouring you started with. (Convince yourself of this.) Similarly, rotating it 3 times will also always give a new colouring (since rotating 3 times is the same as rotating once the other way).

   So, we just need to find the number of these colourings that don't change when rotated twice.

   Note that such a colouring will never have the centre square coloured. If any other square is coloured, then there is a *unique* second square that can be chosen such that when these two squares are coloured, the colouring does not change when rotated twice. (This unique second square is where the first square lands when the board is rotated twice.)

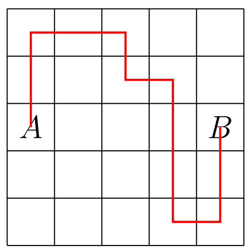
   Apart from the centre square, there are 440 squares. Thus there are 220 colourings that don't change when rotated twice. (The order in which we choose the two squares doesn't matter.)

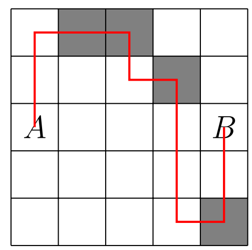
In total, there are $\binom{441}{2}$ ways to choose 2 squares to colour. Out of these, 220 account for 2 colourings under rotation, and the rest account for 4. Thus, the answer is

$$\frac{1}{4}\cdot\left(\binom{441}{2}-220\right)+\frac{1}{2}\cdot 220=\frac{1}{4}\cdot\left(\frac{441\cdot 440}{2}-220\right)+110=\boxed{24310}.$$

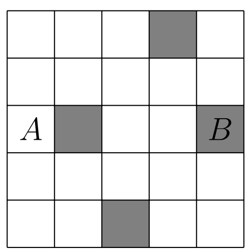
5. Observe that any path from A to B of the required form can be uniquely associated with a choice of 4 squares, one from each of the 2$^{\text{nd}}$, 3$^{\text{rd}}$, 4$^{\text{th}}$, and 5$^{\text{th}}$ columns, where we choose the square in the k^{th} column if and only if it marks the *first* time the path enters that column.

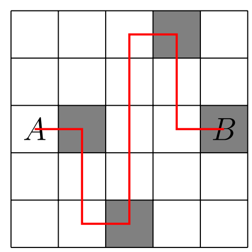
   For example, the path on the left is associated with the squares on the right,





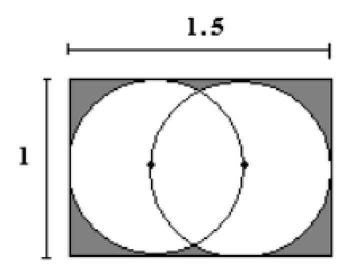
   and the squares on the left uniquely determine the path on the right.





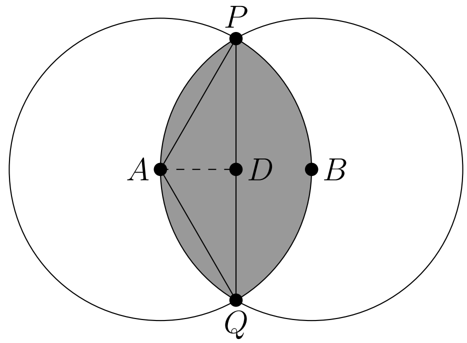
   It follows that the number of paths is equal to the number of ways to choose 4 squares, one from each of the 2$^{\text{nd}}$, 3$^{\text{rd}}$, 4$^{\text{th}}$, and 5$^{\text{th}}$ columns. There are $5^4=\boxed{625}$ ways to do this.

6. To find the area of the shaded region, we will subtract the white area from the area of the rectangle. The white area is equal to twice the area of a circle minus the area of their intersection.



   The radius of a circle is $1.5/3=0.5$, so the area of a circle is $\pi/4$.

   Next, we find the area of the intersection of the two circles. Let A, B be the centres of the two circles and let P, Q be their intersection points, as shown below.



ΔABP and ΔABQ are both equilateral, since their sides are all equal to the radius of a circle. Thus, $\angle PAQ = 60° + 60° = 120°$.

The area of sector $APBQ$ is then

$$[APBQ] = \frac{120}{360} \cdot A_{circle} = \frac{1}{3} \cdot \frac{\pi}{4} = \frac{\pi}{12}.$$

Since $AP = 0.5$ (a radius) and $AD = 0.25$ (half a radius), we have

$$PD = \sqrt{0.5^2 - 0.25^2} = \frac{\sqrt{3}}{4}.$$

Hence, the area of ΔAPQ is

$$[APQ] = PD \cdot AD = \frac{\sqrt{3}}{4} \cdot \frac{1}{4} = \frac{\sqrt{3}}{16}.$$

The area of segment PQB is then

$$[APBQ] - [APQ] = \frac{\pi}{12} - \frac{\sqrt{3}}{16}.$$

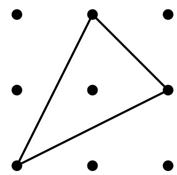
The area of the lens (intersection) is twice this area:

$$A_{lens} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}.$$

Finally, we get that

$$\begin{aligned} A_{shaded} &= A_{rectangle} - 2A_{circle} + A_{lens} \\ &= \frac{3}{2} - \frac{\pi}{2} + \frac{\pi}{6} - \frac{\sqrt{3}}{8} \\ &= \frac{3}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{8}. \end{aligned}$$

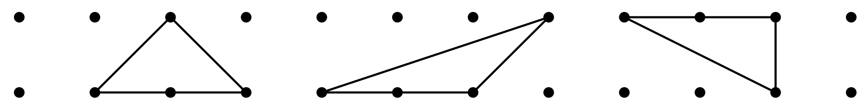
7. We count the triangles in a systematic way. Note first that every triangle with area 1 has either a horizontal side or a vertical side, since the smallest triangle with no horizontal or vertical sides looks like this:



which has area > 1.

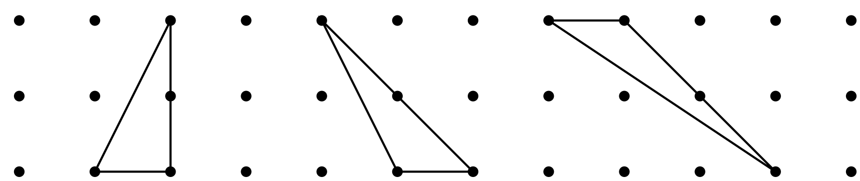
Case 1. Triangles with a horizontal base.

Case 1.1. $base = 2$, $height = 1$. Examples:



If the triangle "points up" (first two figures above), then the base can be on one of the bottom 3 rows. For each row, there are 2 choices for the base. For each base, there are 4 choices for the top vertex (it can be any one of the 4 vertices in the row above). This gives $3 \cdot 2 \cdot 4 = 24$ triangles. Similarly, there are 24 triangles which "point down".
Total: $24 + 24 = 48$.

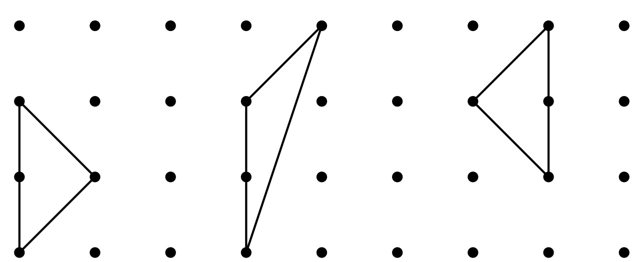
Case 1.2. $base = 1$, $height = 2$. Examples:



If the triangles points up, the base can be on one of the bottom 2 rows. For each row, there are 3 choices for the base. For each base, there are 4 choices for the top vertex. This gives $2 \cdot 3 \cdot 4 = 24$ triangles. Similarly, there are 24 triangles which point down.
Total: $24 + 24 = 48$.

Case 2. Triangles with a vertical base and with **no horizontal side**.

Case 2.1. $base = 2, height = 1$. (Here, the base is vertical and the height is horizontal.) Examples:



If the triangle points to the right, the base can be on one of the 3 left rows. For each row, there are 2 choices for the base. For each base, there are now **2** choices for the right tip. This gives $3 \cdot 2 \cdot 2 = 12$ triangles. Similarly, there are 12 triangles which point to the left. Total: $12 + 12 = 24$.

Case 2.2. $base = 1, height = 2$.
Using the same method, we count 24 triangles for this case.

Thus the total number of triangles with area 1 with vertices on the grid is $48 + 48 + 24 + 24 = \boxed{144}$.

8. Let $E(n)$ denote your expected winnings given optimal play, if you are allowed to roll the die up to n times. For a re-roll to be optimal even if the first roll is a 7, we need the expected earnings from the remaining $n-1$ rolls to exceed 7. That is, we are looking for the smallest n such that $E(n-1) > 7$.

We calculate the values of $E(n)$ one by one.

- $E(1) = 4.5$. (Since you have to stop after the first roll, this is just the average of the numbers from 1 to 8).
- If $n = 2$, you should stop if your first roll is 5 or more (since this beats $E(1) = 4.5$), and re-roll if it's 4 or less (since you'd expect to earn more by rolling one more time). Each case happens with probability $\frac{1}{2}$.
  Thus $E(2) = \frac{1}{2}(6.5) + \frac{1}{2}(4.5) = 5.5$.
  (6.5 is the expected value of a roll given you roll 5 or more; it is just the average of $\{5, 6, 7, 8\}$.)
- $n = 3$: Since $E(2) = 5.5$, we stop if the first roll is 6 or more, otherwise we continue. So $E(3) = \frac{3}{8}(7) + \frac{5}{8}(5.5) = 6.0625$.

Proceeding in this way (and using a calculator), we can recursively calculate $E(k)$ for all k. The table below shows the first values:

k	1	2	3	4	5	6	7
$E(k)$	4.5	5.5	6.06	6.42	6.69	6.89	7.05

The first k with $E(k) > 7$ is $k = 7$. Thus the answer to the problem is $n = \boxed{8}$.

Note. The recurrence formula for $E(k)$ (by the above reasoning), can be written as

$$E(k+1) = \left(\frac{9 - \lceil E(k) \rceil}{8}\right)\left(\frac{8 + \lceil E(k) \rceil}{2}\right) + \left(\frac{\lceil E(k) \rceil - 1}{8}\right) E(k).$$