Part E

1. Since the second digit of an elevated number is equal to its third digit, an elevated number is uniquely determined by its first two digits. So, all we need to do is count the number of ways to choose the first two digits, so that the first digit is strictly less than the second digit. Note that the first digit cannot be 0, since an elevated number is a 3-digit number.

   If first digit is 1: The second digit can be $2, 3, \ldots,$ or 9. $\to$ 8 possibilities.

   If first digit is 2: The second digit can be $3, 4, \ldots,$ or 9. $\to$ 7 possibilities.

   $\vdots$

   If first digit is 8: The second digit must be 9. $\to$ 1 possibility.

   Hence there are $8 + 7 + \cdots + 1 = \boxed{36}$ elevated numbers.

   Alternatively, if you know choose notation, the answer is $\binom{9}{2} = \frac{9 \cdot 8}{2} = \boxed{36}$.

2. By exponent rules, we have

$$20^{15} \times 5^{15} = \left(2^2 \times 5\right)^{15} \times 5^{15} = 2^{30} \times 5^{15} \times 5^{15} = 2^{30} \times 5^{30} = 10^{30}.$$

   Note that $10^{30} = 1\underbrace{000\ldots0}_{30\text{ 0's}}$, which has $\boxed{31}$ digits.

3. Note that there are 9 1-digit numbers, 90 2-digit numbers, and 900 3-digit numbers. If you were to write out all the 1-digit and 2-digit numbers, then you would have written down a total of $9 \times 1 + 90 \times 2 = 189$ digits. On the other hand, if you were to write out all the 1-digit, 2-digit, and 3-digit numbers, then you would have written down a total of $9 \times 1 + 90 \times 2 + 900 \times 3 = 2889$ digits. Since $189 < 2014 < 2889$, the 2014^{th} digit in the very big number must occur within a 3-digit number.

   Now, $2014 - 189 = 1825$, so the 2014^{th} digit in the very big number is the same as the 1825^{th} digit when we write out all the 3-digit numbers in order. Finally, $1825 \div 3 = 608\text{R}1$, so the digit we are looking for is the first digit of the 609^{th} 3-digit number. The 609^{th} 3-digit number is $100 + 608 = 708$, which has first digit $\boxed{7}$.

4. First, note that the volume in the container *not* taken up by water is $10 \times 8 \times 1 = 80$. Thus, when the cube is submerged and the water reaches the top of the container, the volume of the cube submerged in the water is 80. Let d denote the height of the cube above the water. Then $5 - d$ measures the distance from the bottom of the cube to the surface of the water. We have that $5 \times 5 \times (5 - d) = 80$ so $d = 5 - \frac{80}{25} = 1.8$. Hence, the distance from the top of the cube to the bottom of the water container is $d + 6 = \boxed{7.8}$.

5. We use the Triangle Inequality, which says that lengths a, b, c form a triangle if and only if $a + b > c$, $b + c > a$, and $a + c > b$.

   First, we immediately see that if either one of 2015 or 2016 are present as sides in the triangle then the other must also be a side length. This is because no two numbers in

$\{1,2,3,4\}$ can have a sum greater than 2015. Let c be the other side of the triangle. Then, by the triangle inequality, $c + 2015 > 2016$, which implies that $c > 1$, which means that $c \in \{2,3,4\}$.

Next, suppose that 2015 and 2016 are not side lengths of the triangle. Then the side lengths of our triangle must be different numbers from the set $\{1,2,3,4\}$. We claim that 1 cannot be a side length of the triangle. Indeed, if the triangle had lengths 1, a, and b (say with $a < b$), then by the triangle inequality, we would have $1+a > b$, so that $b - a < 1$. Since $a < b$, we also have $0 < b - a$. But then we have $0 < b - a < 1$, which is impossible since a and b are integers. Hence the only possible triple of side lengths in this case is $(2,3,4)$, which does satisfy the triangle inequality and so forms a triangle.

Therefore there are $\boxed{4}$ possible triangles: $(2,3,4)$, $(2,2015,2016)$, $(3,2015,2016)$, and $(4,2015,2016)$.

6. Let the distance between Waterloo and Toronto be d. Since time = distance ÷ speed, Richard takes time $d/80$ on the morning trip and time $d/120$ on the evening trip. His average speed for the entire trip (morning and evening), is:

$$\begin{aligned}\text{average speed} &= \frac{\text{total distance}}{\text{total time}} \\ &= \frac{2d}{d/80 + d/120} \\ &= \frac{2}{1/80 + 1/120} \\ &= \boxed{96}\,(\text{km/h}).\end{aligned}$$

7. First, we must determine the expected payoffs of the two options to save his money. Using compounding of interest rates and expected value of probabilities and payoffs, we derive the following:

Option 1: $\$10,000 \times 1.02^2 = \$10,404$.

Option 2: $75\% \times (\$10,000 \times 1.05^2) + 25\% \times (\$10,000 \times (1 - 0.05)^2) = \$10,525$.

Finally, we take the difference of these two expected payoffs:

$$\$10,525 - \$10,404 = \$\boxed{121}.$$

8. Notice that regardless of what the tens digit is, it does not affect what the units digit will be. Therefore, we may restrict ourselves to finding the units digit of $3^{77} + 7^{33}$.

The units digit of powers of 3 repeat: $3, 9, 7, 1, 3, \ldots$, so the units digit of the powers of 3 form a cycle of length 4.

Similarly, the units digit of the powers of 7 repeat: $7, 9, 3, 1, 7, \ldots$, also forming a cycle of length 4.

Now, $77 \div 4 = 19\,\mathrm{R}\,1$, so 3^{77} has units digit 3 (the first number in the cycle). Similarly, $33 \div 4 = 8\,\mathrm{R}\,1$, so 7^{33} has units digit 7. Thus, the units digit of $33^{77} + 77^{33}$ is the units digit of $3 + 7$ which is $\boxed{0}$.