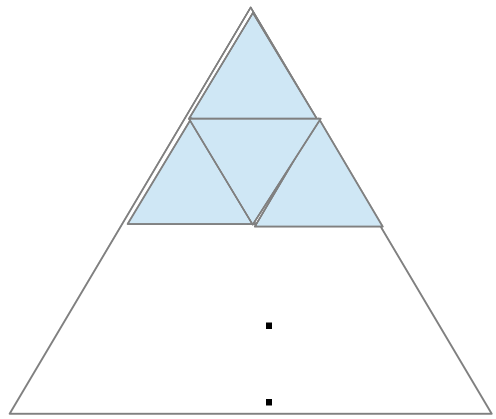
Student Name: ______________________

1. An equilateral triangle of side length 12 is completely filled in by non-overlapping equilateral triangles of side length 1. How many small triangles are required?

There are couple ways of realizing the correct answer here. One way is to actually tile the bigger equaliteral triangle with side length 12 with unit equilateral triangles and realize that there are 12 'rows' of unit triangles, each with 1, 3, 5, ..., 23 triangles respectively:



Summing these up we get **144**.

Once could also notice that the ratio of the areas of these two triangles is essentially the ratios of their side length squared, i.e. $(\frac{12}{1})^2 = 144$. And thus we need 144 unit triangles to cover the whole area of the triangle with side length 12.

2. What is the probability that an integer in the set $\{1, 2, 3, \ldots 100\}$ is divisible by 2 and not divisible by 3?

Let's first count the number of integers in the given set that is either divisible by 2 and not divisible by 3. First of all there are $\frac{100}{2} = 50$ integers divisible by 2. Out of these 50 numbers, those divisible by 3 will also be divisible by 6. Thus we need to subtract off $\frac{100}{6} = 16$ reminder 4. Thus there are 50 – 16 = 34 numbers in the given set that are divisible by 2 but not 3. Since there are a total of 100 numbers in the set, the required probability is $\frac{34}{100} = \boxed{\frac{17}{50}}$.

3. Suppose that the number α satisfies the equation $4 = \alpha + \alpha^{-1}$. What is the value of $\alpha^4 + \alpha^{-4}$?

The trick to solving this problem is to repeatedly square both sides. The given equation is $4 = \alpha + \frac{1}{\alpha}$. Squaring both sides yields $16 = \left(\alpha + \frac{1}{\alpha}\right)^2 = \alpha^2 + 2\alpha\left(\frac{1}{\alpha}\right) + \frac{1}{\alpha^2} = \alpha^2 + \frac{1}{\alpha^2} + 2$. Subtracting 2 from both sides, we get $14 = \alpha^2 + \frac{1}{\alpha^2}$. Now, we square both sides again!

Student Name: ______________________

$14^2 = 196$, so $196 = \left(\alpha^2 + \frac{1}{\alpha^2}\right)^2 = \alpha^4 + 2(\alpha^2)\left(\frac{1}{\alpha^2}\right) + \frac{1}{\alpha^4} = \alpha^4 + \frac{1}{\alpha^4} + 2$. Subtracting 2 from both sides again, we get $\alpha^4 + \frac{1}{\alpha^4} = \boxed{194}$.

4. Alice flips 8 coins, 3 of which are quarters and 5 of which are pennies. What's the probability that an odd number of pennies come up heads AND the total number of heads (out of all coins) is even?

For an odd number of pennies to come up heads **and** the total number of heads to be even, an odd number of quarters must also come up heads. Let's consider the quarters first. Since there are 3 quarters, either 1 or 3 of them must come up heads. Since it is equally likely to toss a head as to toss a tail, the probability of 1 or 3 quarters coming up heads is really the same as the probability of 1 or 3 quarters coming up tails. But 1 or 3 quarters coming up tails means that 0 or 2 quarters come up heads! So the probability of 1 or 3 quarters coming up heads is **exactly the same** as the probability of 0 or 2 quarters coming up heads. Since you can only come up with 0, 1, 2 or 3 heads when tossing 3 coins, the probability of coming up with 1 or 3 heads is $\frac{1}{2}$. A similar argument holds for the 5 pennies – the probability of tossing an odd number of heads is again $\frac{1}{2}$. The answer is $\frac{1}{2} \times \frac{1}{2} = \boxed{\frac{1}{4}}$ or $\boxed{0.25}$ or $\boxed{25\%}$.

In fact, we can generalize this argument to n coins, where n is odd. The probability of tossing an odd number of heads is equal to the probability of tossing an odd number of tails, since heads and tails are equally likely. But tossing an odd number of tails is the same as tossing an even number of heads, since the total number of coins is odd. So it is equally likely to toss an odd number of heads as to toss an even number of heads, and both these probabilities are equal to $\frac{1}{2}$.

5. The table below shows an example of "Suffix Code". Each letter is represented by a variable length binary sequence "codeword". To send a message, you simply put together the codewords for your word. For example, "DOG" would be encoded as "011011100001". However, decoding a message is tricky because you don't know how the letters are separated. It is called a "Suffix Code" because no codeword is the suffix of another codeword. This property allows for an unambiguous decoding of each message. Please decode the message:

11100010100110101110001100010

Student Name: ______________________

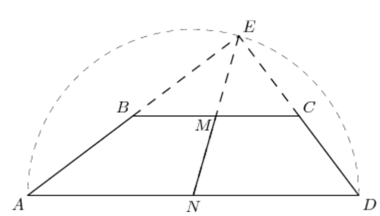
character	codeword	character	codeword
!	000	U	01001
B	0100	N	11001
S	01100	C	0101
R	11100	K	1101
E	010	F	00011
D	0110	V	10011
O	1110	I	1011
G	0001	A	111

If you decode it from left to right you will encounter a series of ambiguities that are difficult to solve, if not impossible. For example first 3 digits 111 match 'A', but first 5 digits match 11100 is 'R', etc. Since the codewords of the suffix code each has a unique suffix, decoding it backwards from right to left would be straight forward. Since each character is coded with codeword of 3 or more digits, we check last 3 digits, 010, which matches character 'E', no other codewords has a suffix of 010, so the last character of the message is decoded as 'E'; similarly, we can decode the message to be **REVERSE**.

6. Consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for k ≥ 1.
Given that $a_m + a_{m+1} + \ldots + a_{2012} = \frac{10}{2013}$ for positive integer m < 2012, find m.

Note that $a_k = \frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}$ (do the algebra to convince yourself of this). This may seem complicated at first, but it will make our lives a lot easier! Using $a_k = \frac{1}{k} - \frac{1}{k+1}$, we get that $a_m + a_{m+1} + \cdots + a_{2012} = \left(\frac{1}{m} - \frac{1}{m+1}\right) + \left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \cdots + \left(\frac{1}{2012} - \frac{1}{2013}\right)$. Can you see why we used the new expression for a_k now? The middle terms in the sum all cancel out, leaving us with $\frac{1}{m} - \frac{1}{2013}$. We are told that this is equal to $\frac{10}{2013}$, so $\frac{1}{m} - \frac{1}{2013} = \frac{10}{2013}$, or $\frac{1}{m} = \frac{11}{2013}$. Thus, $m = \frac{2013}{11} = \boxed{183}$. (Note: The trick we used is called a "telescoping sum".)

7. In trapezoid $ABCD$ with BC parallel to AD, let $BC = 14$ and $AD = 38$. Let $\angle A = 39°$, $\angle D = 51°$, and M and N be the midpoints of BC and AD, respectively. Find the length MN.



Student Name: ______________________

Extend AB and DC to meet at a new point E outside the trapezoid. Consider triangle AED. Note that $\angle AEB$ = 180° - $\angle A$ - $\angle D$ = 90°. Therefore, AD is a hypotenuse and its midpoint, N, is the centre of the circle passing through A, E, and D. Therefore EN = 19. Note that EN passes through M, since BC is parallel to AD. By similar triangles EBM and EAN, we have $\frac{BM}{AN} = \frac{EM}{EN} = \frac{7}{19}$, therefore EM = 7. Thus MN = EN - EM = 19 - 7 = $\boxed{12}$.

8. In this problem, we consider the experiment of rolling a set of dice and summing the numbers to get an outcome. Two sets of dice are said to be equivalent if the probability of rolling every outcome is equal in both sets. Consider the following set of dice:

- three 12-sided dice with faces 4, 4, 4, 4, 2, 2, 2, 2, 0, 0, 0, 0
- an 8-sided die

What do the faces of the 8-sided die need to be for this set of dice to be equivalent to the set of three regular 6-sided dice (with faces 1, 2, 3, 4, 5, 6)?

The maximum value with three regular 6-sided dice is 18. Therefore, the maximum face on the 8-sided die must be 6 (so that 18 can be attained if each of the 12-sided dice land on a 4). The probability of rolling an 18 with three 6-sided dice is $\frac{1}{216}$. The probability of rolling a 12 with the three 12-sided dice is $\frac{1}{27}$, so the probability of rolling a 6 with the 8-sided die must be 1/8, since $(\frac{1}{27})\ (\frac{1}{8}) = \frac{1}{216}$. Thus exactly one face is 6.
Similarly, the minimum value with three 6-sided dice is 3. Therefore, the minimum face on the 8-sided die must be 3, and exactly one face is 3.
So all remaining faces are either 4 or 5. We would like to argue that the number of 4's must be equal to the number of 5's. The intuitive explanation is that for the three 6-sided dice, the result is symmetric in-between the lowest possible outcome (3) and the highest possible outcome (18). Since the 12-sided dice are also symmetric in this way, the 8-sided die must also be symmetric.

The technical explanation is that on average (in expected value), each 6-sided die gives a value of 3.5, so the total expected value is 10.5. Each 12-sided die has an expected value of 2. Therefore, the expected value of the 8-sided die must be 10.5 – 3× 2 = 4.5. So it must have an equal number of 4's and 5's.
The answer is $\boxed{\mathbf{3, 4, 4, 4, 5, 5, 5, 6}}$.