Student Name: ______________________

1. What is the sum of all the natural numbers from 250 to 750 inclusive?

Recall that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$. The sum of the numbers from 250 to 750 can be thought of as the sum of the numbers from 1 to 249 subtracted from the sum of the numbers from 1 to 750. Thus the sum is $\frac{750(751)}{2} - \frac{249(250)}{2} = 250500$

2. Determine the number of divisors of 15! ($15! = 15 \times 14 \times \ldots \times 1$) that are also perfect squares.

Writing out the prime factorization of 15! we get: $2^{11} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13$ Any divisor of 15! will have a prime factorization that is a subset of 15! (i.e. primes amongst the ones in 15!'s prime factorization) and the exponents of the primes won't exceed the exponents in 15!'s prime factorization (e.g. any divisor of 15! won't have and power of 2 greater than 2^{11}). For this divisor to be a perfect square, we need each of these exponents to be even. Thus the possible exponents for 2 in any divisor of 15! that is also a perfect square are 0, 2, 4, 6, 8, 10 (i.e. the 6 even numbers from 0 – 11). Doing this for each prime, we see that there are a total of $6 \times 4 \times 2 \times 2 \times 1 \times 1 = 96$ divisors of 15! that are also perfect squares.

3. Determine the unit's digit of 2012^{2011}.

The unit's digit of a number is the remainder you get when you divide that number by 10 (i.e. that number modulo 10). Thus we need to calculate $2012^{2011} \equiv 2^{2011}$ (mod 10). Calculating the powers of 2 modulo 10, we get that:

$$2^1 \equiv 2 \pmod{10}$$
$$2^2 \equiv 4 \pmod{10}$$
$$2^3 \equiv 8 \pmod{10}$$
$$2^4 \equiv 6 \pmod{10}$$
$$2^5 \equiv 2 \pmod{10}$$

This pattern repeats every 4 powers of 2, thus we take 2011 modulus 4 and get $2011 \equiv -1 \equiv 3 \pmod 4$. Thus $2^{2011} \equiv 8 \pmod{10}$, and so the last digit of 2012^{2011} is 8.

4. How many real number solutions does $x^2 + \sqrt{x^4 + 3} = 1$ have?

Rearranging the equation we get $\sqrt{x^4 + 3} = 1 - x^2$. Squaring both sides we get $x^4 + 3 = (1 - x^2)^2 = x^4 - 2x^2 - 1$. Thus $x^4 + 3 = x^4 - 2x^2 - 1$, giving us that $x^2 = -2$. However since x is real, this isn't possible. Thus the given equation has no solutions.

5. Grace walks up an escalator that is moving up. When she walks at 1 step per second, she takes 20 steps. When she walks at 2 steps per second, she takes 32 steps. Assuming that Grace never skips a step and that the speed of the escalator is constant, how many steps

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does the escalator have?

Let x represent the total number of steps the escalator has. When she walked at 1 step/s she took 20 steps; therefore when she stepped off the escalator, the starting step (i.e. the step she entered on) is 20 steps behind her (since she only took 20 steps). Thus the escalator travelled x – 20 steps in 20s. Similarly, since she walked 32 steps when she walked at the speed of 2 steps/s, this the escalator travelled x – 32 steps in 16s. Since the speed of the escalator is constant, we have that $\frac{x-20}{20s} = \frac{x-32}{16s}$, and thus x = 80 steps.

6. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2do not divide n?

Each divisor of n is of the form 2^a3^b, where 0 <= a<= 31 and 0 <= b<= 19, and so there are $32 \times 20 = 640$. Similarly, all the divisors of n^2 are of the form 2^c3^d, where 0 <= c<= 62 and 0 <= d<= 38, and so there are $63 \times 39 = 2457$. Since every divisor of n is also a divisor of n^2, thus the number of divisors of n^2that do not divide n is 2457 – 640 = 1817.

7. Larrylives on the xy plane and he owns a pogo stick. He can jump 3 or 7 steps to the right, or 5 steps up. How many possible ways are there for Larry to go from the origin (0, 0) to (25, 25)? (You can leave your answer in terms of factorials)

We find the number of jumps to the right Larry must take. To do that, we find the number of non-negative solutions to $3x + 7y = 25$.Since $0 \leq \ y \leq 3$, we simply try all value values of y to see that the only solution is (x, y) = (6, 1). So if we were to represent each jump upwards with A, a jump to the right of length 3 with B, and a jump to the right of length 7 with C, then the number of ways for Larry to get to (25, 25) is the same as the number of strings containing 5 A's, 6 B's and 1 C. The 5 A's can occur in $\binom{5+6+1}{5}$ possible positions, and the 6 B's can occur in $\binom{6+1}{6}$ possible positions. Therefore, there are $\binom{12}{5}\binom{7}{6} = \frac{12!}{5!6!1!} = 5544$ possible ways.

8. What is the edge length of the largest regular tetrahedron that fits inside a $1 \times 1 \times 1$ cube?

First of all, notice that the largest regular tetrahedron can be aligned in the cube so that one of its edgeslays on a face of the cube (i.e. you can do this by rotating the tetrahedron inside the cube). Now notice that the maximum length of this edge of the tetrahedron on this face is $\sqrt{2}$ (i.e. the diagonal of the face). In fact, choosing 4 of the vertices of the cube, one can form the largest regular tetrahedron. Thus the edge length of the largest regular tetrahedron that one can fit inside a $1 \times 1 \times 1$ cubeis$\sqrt{2}$.