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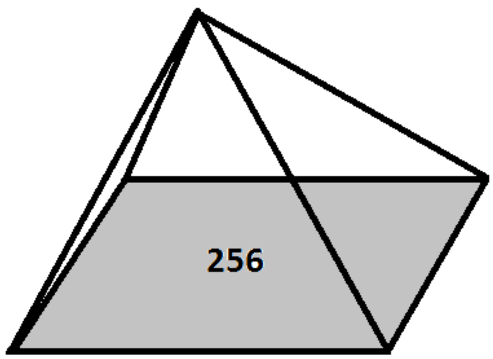
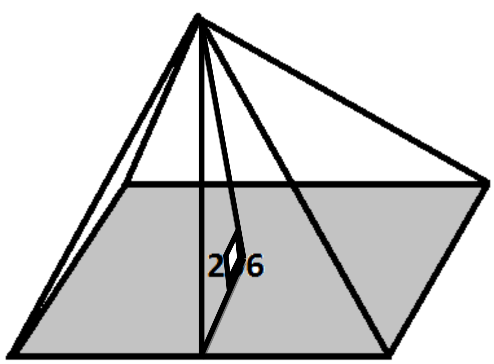
1. One day Cindy said, "I have been alive for all or part of four decades." Rounded to the nearest year, what is the youngest Cindy could have been?

If Cindy is born in the second half of a decade, then in 20 years (rounded to the closest year) she would have been alive for part of 4 decades. Any less than this and it is impossible for this to be true. Thus the youngest Cindy could have been rounded to the nearest year is 20 years old.

2. A square based pyramid is formed by one square, and four equilateral triangles. If the area of the square (the base) is 256, what is the height of the pyramid?



The side length of the square base, and by extent the side length of the equilateral triangle faces, is 16 (since the area is 256). Notice that the height of the pyramid, the height of any one of the triangle faces and half of the square bass forms a right angled triangle with the height of the equilateral triangle being the hypotenuse:



Since the side length of the equilateral triangle is 16, thus it has a height of $8\sqrt{3}$. Thus, if h is the height of the pyramid, then by Pythagoras' theorem, we have that $\left(\frac{16}{2}\right)^2 + h^2 = \left(8\sqrt{3}\right)^2 = 192$.Thus $h = 8\sqrt{2}$, and so the height of the pyramid is $8\sqrt{2}$.

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3. Find the smallest positive integer that leaves a remainder of 1 when divided by 2, 3, 4, 5, and 6.

The intended answer was 61 (which is 1 more than the lowest common multiple of 2, 3, 4, and 5), however technically the smallest such number is 1. Marks were given for both 1 and 61

4. Bob wants to write down some of the natural numbers from 1 to 10 in strictly increasing order. For example, Bob could have only written 2, or he could have written 3, 4, 9, 10. How many possible sequences of numbers could he have written (Note: Bob has to write down at least one of the numbers from 1 to 10)?

Bob can either choose to include a particular number or not (as the order he writes the numbers in is fixed). Thus for each of the numbers from 1 to 10 he has 2 choices (write it or not to write it). Thus he has a total of $2^{10} = 1024$ choices. However, since he can't not choose all the numbers (as he has to write down at least one of the numbers) he in fact only has 1023 choices.

5. Write $1 \times 2 \times 3 \times ... \times 15$ as a product of prime powers.

Counting the number of factors of 2, 3, 5, 7, 11, and 13 there are in the numbers from 1 to 15 (we need not go any higher, as the next prime number is greater than 15), we see that $1 \times 2 \times 3 \times ... \times 15 = 2^{11} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13$.

6. How many natural numbers less than 1000 contain at least one 9?

Since its easier to count the number of numbers less than 1000 that don't contain a 9, we do so and then subtract this from 1000 to get the number of numbers less than 1000 that contain at least one 9. You can think of a number less than 1000 as ABC, where A, B and C are digits (i.e 0, 1, 2, …, 9). Since we don't want either of them to be a 9, we have 9 choices for each of them, giving us a total of $9^3 = 729$ numbers less than 1000 without a 9. Thus the number of numbers less than 1000 that contain at least one 9 is $1000 - 729 = 271$.

7. Brian shuffled a standard deck of cards (without the jokers) and randomly drew 4 cards. What is the probability that he drew at least one Ace? (A standard deck of cards contains 52 cards)

As in the previous question, it is easier to calculate count the probability of drawing no aces and subtracting this from 1. Since the total number of ways we can draw 4 cards from the 52 cards in the deck is $\binom{52}{4}$, and the total number of ways to choose 4 cards from the 48 non-Ace cards is $\binom{48}{4}$, thus the probability of drawing no Ace is $\frac{\binom{48}{4}}{\binom{52}{4}}$. Therefore the

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probability of drawing at least one Ace is $1-\frac{\binom{48}{4}}{\binom{52}{4}}=\frac{15229}{54145}$.

8. Three decimal numbers x, y and z are randomly chosen between 0 and 1 inclusive. What is the probability that$x^2+y^2+z^2 \leq 1$?

Notice that three decimal numbers x, y and z chosen between 0 and 1 inclusive form a point on the 3D coordinate plane within the unit cube from (0, 0, 0) to (1, 1, 1). Furthermore, the points whose coordinates follow the inequality $x^2+y^2+z^2 \leq 1$ are within the portion of the unit sphere centered at the origin in the unit cube mentioned before. Thus the probability of the randomly chosen point in the cube being in this portion of the sphere is the ratio of the volume of this portion of the sphere to the volume of the unit cube. This portion of the sphere is exactly $\frac{1}{8}$th of the sphere (since it lays in the positive x, y and z axis of the 3D plane). Thus the probability is $\frac{V_{\frac{1}{8}th\ of\ sphere}}{V_{cube}}=\frac{\frac{\frac{4}{3}\pi 1^3}{8}}{1^3}=\frac{\pi}{6}$.