1. What is the smallest positive integer such that, if we remove the leftmost digit, the resulting number is 1/33 of the original integer?

Answer: 825

* *Clearly the number cannot be 1-digit. If the number was 2-digit, then let the digits be AB. We have* $B = \frac{1}{33}(10A + B)$

$33B = 10A + B$

$16B = 5A$

$B = \frac{5}{16}A$

This means, to make B a positive integer, A is at least 16, which cannot be, since A is a digit (so it is from 0 to 9).

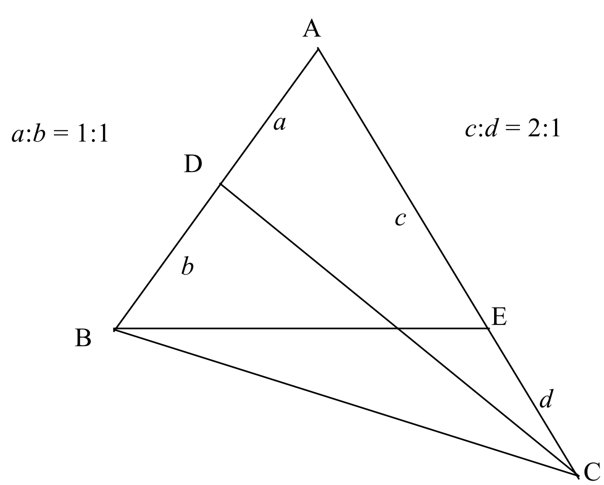
So let's now try 3-digit numbers. Let the number be 100a + b where $1 \le a \le 9$, $0 \le b \le 99$. *We have* $b = \frac{1}{33}(100a + b)$

$33b = 100a + b$

$8b = 25a$

Thus a = 8, b = 25 works. So 825 is the answer.

2. D divides AB in half (a:b = 1:1). E divides AC in the ratio of 2:1, with AE being twice as long as EC (c:d = 2:1). The area of triangle ABE is 1. What is the area of triangle ACD?



Answer: 3/4

* *We use the fact that if two triangles have the same base, their areas are proportional to their heights.*

ABC has the same base AB as ABE, but 3/2 the height since AC:AE = 3:2. Thus ABC has area 3/2.
ABC and ADC have same base AC, but half the height, since AD:AB = 1:2. Thus ADC has area 3/4.

3. For a positive whole number n, define $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots 2 \cdot 1$. For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, while $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. How many zeros are at the end of 100! ?

Answer: 24

* *To make a zero, we need a factor of 2 and a factor of 5. 100! Will have more 2's and 5's, so we only care about how many factors of 5 it has. Each multiple of 5 gives a factor of 5 to 100!, while the 4 multiples of 25 (25, 50, 75, 100) give an extra factor of 5. There are 100 / 5 = 20 multiples of 5 from 1 to 100. So in total there 20 + 4 = 24 factors of 5. So there are 24 zeros at the end of 100!.*

4. Isaac and Carl play a tennis match, where the first person to win 3 sets wins the match. Assume that the players are equally skilled, so that each player is equally likely to win each set. Let x be the probability that the match finishes in exactly 4 sets. Let y be the probability that the match finishes in exactly 5 sets. Compute x - y.

Answer: 0

* Solution 1: *We can think of a match as a word with only letters I and C, where I means Isaac won the set, and C means Carl won the set. For example, ICII means Isaac won sets 1, 3, 4 with Carl winning set 2, and the match finishing in 4 games. The number of ways for the match to finish in exactly 4 games requires each player to win at least 1 of the first 3 games. So the possibilities are*

ICII CICC IICI CCIC CIII ICCC

The number of ways for the match to finish in exactly 5 games requires each player to win exactly 2 of the first 4 games. The number of ways this could happen is $\binom{4}{2} = 6$, *with the possibilities being*

IICC CIIC ICIC CICI ICCI CCII

Thus the probability of the match finishing in 4 games is equal to the probability of the match finishing in 5 games. So $x - y = 0$.

Solution 2: *Assume that the match doesn't finish in 3 sets, so it finishes in either 4 or 5 sets. So after 3 sets, one person is ahead 2-1. If that person wins, the match finishes in 4 sets. Otherwise, the match finishes in 5 sets. Therefore, the chances that the match finishes in 4 sets is equal to the chances that the match finishes in 5 sets. So* $x - y = 0$.

5. In Mike's aquarium, the ratio of gold fish to guppies is 5:4. After Mike buys 18 guppies, the ratio of gold fish to guppies is now 4:5. How many guppies did Mike originally have?

Answer: 32

* *Let Mike originally have x gold fish and y guppies,* $\frac{x}{y} = \frac{5}{4}$. *So* $4x = 5y$.

But $\frac{x}{y+18} = \frac{4}{5}$ *or* $5x = 4y + 72$

$20x = 16y + 4 \cdot 72$

$5 \cdot 5y = 16y + 4 \cdot 72$

$9y = + 4 \cdot 72$

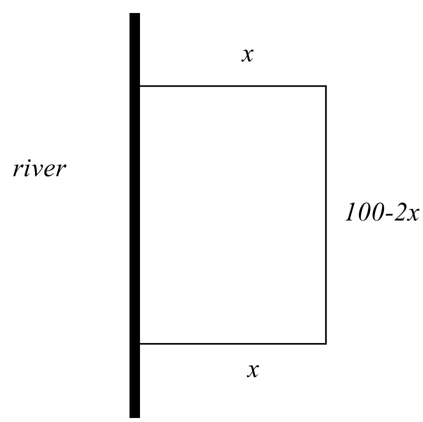
$y = 32$

Mike originally had 32 guppies.

6. A farmer has 100 meters of fencing and he wants to construct a rectangle to fence off as much area as possible. Fortunately, there is a river in his fields (in the shape of a straight line) that he can use as one side of the fence. What is the maximum area he can surround using the fence and the river?

Answer: 1250 m^2

* *Let x be the length of the fence perpendicular to the river. So the length of the fence parallel to the river is 100 - 2x, and the enclosed area is* $x(100 - 2x) = 100x - 2x^2$. *This is a downward opening parabola. Its maximum occurs at x = 25 since its roots are 0 and 50. Thus the maximum area he can fence off is* $25\ (100 - 2 \times 25) = 25 \times 50 = 1250\ m^2$.



7. Let's call a number 'bizarre' if it is a multiple of 9, but the sum of its digits is not 9. How many bizarre numbers are there from 1 to 1000?

Answer: 56

* *The digits must sum to a multiple of 9, by the divisibility rule for 9. Thus they must sum to 18 or 27; they cannot sum to larger than 27.*

<u>Sum 18</u>: Let the number be ABC (1000 does not work so we can assume the number is 3 digits or less).

- *A = 0: B = C = 9 is the only possibility.*
- *A = 1: B = 9, C = 8 or vice versa, so 2 possibilities.*
- *A = 2: B = 9, 8, or 7, so 3 possibilities.*
- *A = 3: B = 9, 8, 7, or 6, so 4 possibilities.*
- *A = 4: B = 9, 8, 7,6, or 5, so 5 possibilities.*
- *A = 5: B = 9, … or 4, so 6 possibilities.*
- *A = 6: B = 9, … or 3, so 7 possibilities.*
- *A = 7: B = 9, … or 2, so 8 possibilities.*
- *A = 8: B = 9, … or 1, so 9 possibilities.*
- *A = 9: B = 9, … or 0, so 10 possibilities.*

<u>Sum 27</u>: Only possibility is 999.

∴ In total there are (1+ 2 + … + 9 + 10) + 1 = 56 numbers.

8. What is the remainder when

$$2^{9^{9}} + 3^{2^{2}}$$

is divided by 7?

Answer: 5

* *The remainders when powers of 2 are divided by 7 follow the pattern 2, 4, 1, 2, 4, 1, …. Since* $9^{9^{9}}$ *is a multiple of 3, the remainder will be 1.* $3^{2^{2}} = 3^4 = 81$*; the remainder when 81 is divided by 7 is 4. So* $2^{9^{9^{9}}} + 3^{2^{2}}$ *is divided by 7 is 1 + 4 = 5.*