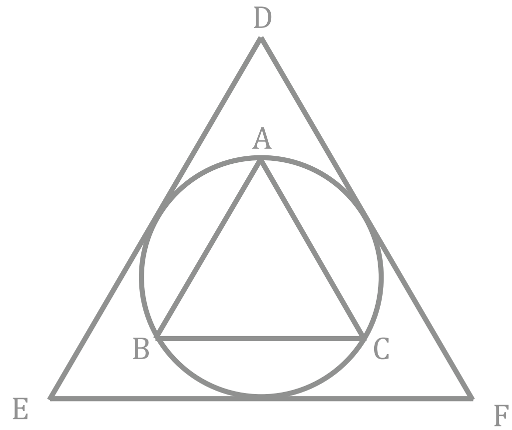
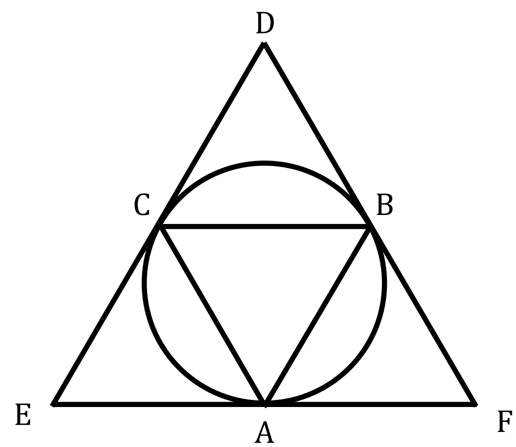
1. An equilateral triangle ABC is inscribed inside a circle, which is inscribed inside an equilateral triangle DEF. Determine the ratio of the area of triangle DEF to the area of triangle ABC.



Solution: The easiest way to solve this problem is to flip the middle triangle. This does not change its area:



Now it is clear that the large equilateral triangle is split into 4 identical equilateral triangles, each with 1/4 of the area. Therefore the ratio of the area of triangle DEF to the area of triangle ABC is **4**.

2. A palindrome is a number that is equal to itself when its digits are read from left-to-right instead of from right-to-left. For example, the number 53235 is a palindrome. How many five-digit palindromes are there whose digits add up to an even number?

Solution: Suppose the digits of a 5-digit palindrome are abcba. Then the sum of the digits of the palindrome would be a+b+c+b+a=2a+2b+c. Since 2a and 2b are always even, the sum of the digits of the palindrome is even if and only if c is even. Therefore, we must simply count the number of 5-digit palindromes with even middle digit. The first digit can be any non-zero digit, so there are 9 choices for the first digit. The second digit can be anything, so it has 10 choices. The third digit must be even, so it has 5 choices. The fourth and fifth digits must equal the second and first, so there is only one choice for those. Hence the total number of choices is 9x10x5=**450**.

3. There are 64 people, numbered 0, 1, 2, …, 63. Each person is either a truth-teller (who always tells the truth) or a liar (who always lies). If n is even, then person n says, "Person n/2 is a truth-teller". If n is odd, then person n says, "Person n-1 is a liar". Given that person 0 is a truth-teller, find the only liar whose number is divisible by 6.

Solution 1: Observe that every odd number n has the opposite truth-value of n-1, and every even number n has the same truth-value as the number n/2. In other words, multiplying by 2 leaves the truth-value the same, while adding 1 to an even number flips the truth-value.

We check all the multiples of 6 to see if they are liars. 0 is a truth-teller. 1 is a liar, and hence 2 is a liar and 3 is a truth-teller. This means 6, 12, 24, and 48 are all truth-tellers. To check 18, we notice that 18=2x9, and 9=8+1. Since 2 is a liar, so are 4 and 8. Hence 9 is a truth-teller, so 18 and 36 are truth-teller. Next, we check 30. Notice that 30=2x15, and 15=1+14 with 14=2x7. Since 6 is a truth-teller, 7 is a liar, so 14 is a liar. This means 15 is a truth-teller, so 30 and 60 are truth-tellers. The only numbers left to check are 42 and 54. Proceeding in the same manner, it is easy to verify that 42 is a liar, while 54 is a truth-teller. Hence the answer is **42**.

Solution 2: Consider the numbers written in binary. If a number ends in 0, then it is even, so dividing by 2 preserves the truth-value. Hence removing 0's from the end of the number preserves the truth-value. If the number ends in a 1, then changing the last digit to a 0 flips the truth-value. It follows that the truth-value of a number only depends on the number of 1's in its binary representation. Indeed, a number is a truth-teller if its representation has an even number of 1's, and a liar otherwise.

In base 2, a number is divisible by 2 if and only if its last digit is 0, and a number is divisible by 3 if and only if the quantity of 1's in even positions minus the quantity of 1's in odd positions is a multiple of 3 (this is similar to the divisibility rule for 11 in base 10). A number is divisible by 6 if it satisfies both conditions. We are looking for a number that is divisible by 6, has an odd number of 1's, and has at most 6 digits. The only such number is 101010, which is the binary representation of 42.

4. I have an arithmetic sequence of 25 numbers. The sum of the numbers in the sequence is 100. What is the 13th number?

Solution: The 13th number is the middle number of the sequence. Since the sequence is an arithmetic sequence, the middle number is equal to the average of all the numbers in the sequence. We know the sum is 100, and there are 25 numbers. This means the average is 100/25=4, so the 13th number of the sequence is **4**.

5. You roll 10 dice. What is the probability that the sum of the numbers on the 10 dice is divisible by 3?

Solution: Consider the last die rolled. The sum of the first 9 dice is one of 0, 1, or 2 modulo 3. The last die has equal probability of being 0, 1, or 2 modulo 3. Therefore, if the sum of the first 9 dice is 0 modulo 3, the last die has 1/3 chance of resulting in 0 modulo 3, which gives a sum divisible by 3. Similarly, if the sum of the first 9 dice is 1 modulo 3, the last die has 1/3 chance of resulting in 2 modulo 3, and if the sum is 2 modulo 3, the last die has a 1/3 chance of resulting in 1 modulo 3. In all cases, the last die has a 1/3 chance of landing on a number that gives an overall sum divisible by 3. Since this is true regardless of what happens with the first 9 dice, we conclude that the probability that the sum of all 10 dice is divisible by 3 is **1/3**.

6. Evaluate $\left(\frac{1}{2}\times\frac{1}{3}\right)+\left(\frac{1}{3}\times\frac{1}{4}\right)+\left(\frac{1}{4}\times\frac{1}{5}\right)+\cdots+\left(\frac{1}{99}\times\frac{1}{100}\right)$.

Solution: Notice that $\frac{1}{n}\times\frac{1}{n+1}=\frac{1}{n}-\frac{1}{n+1}$. Therefore, the expression becomes

$$\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)$$

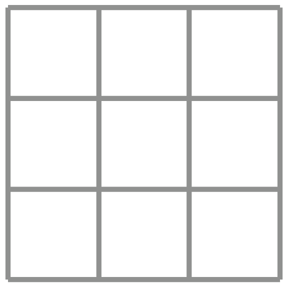
Consecutive terms of this sequence cancel with each other, so this is equal to

$$\frac{1}{2}-\frac{1}{100}$$

So the answer is **49/100** or **0.49**.

7. A 3x3 grid has a number written in each of the 9 cells, with the following properties:
   a. The product of the numbers in any row is 1
   b. The product of the numbers in any column is 1
   c. The product of the numbers in any 2x2 grid is 5

   What is the value of the middle number in the grid?



<u>Solution</u>: There are 4 different 2x2 grids inside the 3x3 grid. If we multiply the numbers in all 4 of these grids, we get 5x5x5x5=625. Inside this product, the corners of the grid each occur once, the centre occurs 4 times, and the remaining cells occur twice each. If we multiply the numbers in each of the three columns, and then multiply the result by the product of the middle column and the product of the middle row, we get 1x1x1x1x1=1. Inside this product, the corners occur once each, the centre occurs 3 times, and the remaining cells occur twice each.

Dividing the first product by the second gives 625. The result is simply the value of the centre of the grid. Therefore, the centre of the grid is <u>**625**</u>.

8. Jim only keeps three kinds of coins in his wallet: pennies, nickels, and dimes. How many ways are there for Jim to have 10 coins in his wallet?

<u>Solution</u>: We can represent a wallet in the following form: a "0" denotes a coin, and a "1" denotes a shift in the types of coins. In this way, we will use 00101000 to represent 2 pennies, 1 nickel, and 3 dimes (since there are 2 zeros before the first "1", 1 zero between the two "1"s, and 3 zeros after the second "1"). As another example, 110 will represent a wall with one dime only, and 11 will be the empty wallet. We will always use exactly two "1"s when representing a wallet.

It is not difficult to see that the wallets with 10 coins correspond exactly to strings of zeros and ones with 10 zeros and 2 ones. Our task then becomes counting these strings. There are 12 digits in each string, and 2 of those digits are "1". Therefore, there are $\binom{12}{2}$ ways of choosing where to place the "1"s inside the string. This means there are $\binom{12}{2} = 66$ possible strings, so there are <u>**66**</u> ways for Jim to have 10 coins in his wallet.